

Guiding Principles
and Key Components
of an Effective
Mathematics
Program

$$1 + 2 = 3$$

$$8 + () = 13$$

A long-standing content issue in mathematics concerns the balance between theoretical and applied approaches. Mathematics is both. In the theoretical (pure) sense, mathematics is a subject in its own right with distinct methods and content to be studied. But mathematics is also extremely applicable both in the practical sense and in connection to other realms of study, including the arts, humanities, social sciences, and the sciences. Any comprehensive representation of the content of mathematics must balance these aspects of beauty and power.

—A. Holz, *Walking the Tightrope*

The major goals of mathematics education can be divided into two categories: goals for teachers and goals for students.

Goals for Teachers

Goals for teachers to achieve are as follows:

1. Increase teachers' knowledge of mathematics content through professional development focusing on standards-based mathematics.
2. Provide an instructional program that preserves the balance of computational and procedural skills, conceptual understanding, and problem solving.
3. Assess student progress frequently toward the achievement of the mathematics standards and adjust instruction accordingly.
4. Provide the learning in each instructional year that lays the necessary groundwork for success in subsequent grades or subsequent mathematics courses.
5. Create and maintain a classroom environment that fosters a genuine understanding and confidence in all students that through hard work and sustained effort, they can achieve or exceed the mathematics standards.
6. Offer all students a challenging learning experience that will help to maximize their individual achievement and provide opportunities for students to exceed the standards.
7. Offer alternative instructional suggestions and strategies that address the specific needs of California's diverse student population.
8. Identify the most successful and efficient approaches within a particular classroom so that learning is maximized.

Goals for Students

Goals for students to achieve are as follows:

1. Develop fluency in basic computational and procedural skills, an understanding of mathematical concepts, and the ability to use mathematical reasoning to solve mathematical problems, including recognizing and solving routine problems readily and finding ways to reach a solution or goal when no routine path is apparent.

2. Communicate precisely about quantities, logical relationships, and unknown values through the use of signs, symbols, models, graphs, and mathematical terms.
3. Develop logical thinking in order to analyze evidence and build arguments to support or refute hypotheses.
4. Make connections among mathematical ideas and between mathematics and other disciplines.
5. Apply mathematics to everyday life and develop an interest in pursuing advanced studies in mathematics and in a wide array of mathematically related career choices.
6. Develop an appreciation for the beauty and power of mathematics.

California students have made gains recently; however, they continue to lag behind students in other states and nations in their mastery of mathematics (NCES 2003; Reese et al. 1997; Beaton et al. 1996). Comparing the 1990s with the 1970s, a study found that the number of students earning bachelor's and master's degrees in mathematics decreased during those 20 years (NCES 1997). At the same time the number of students entering California State University and needing remediation in mathematics has been increasing (California State University 1998). The result of students achieving the goals of this framework and mastering the California mathematics standards will be not only an increase in student mastery of mathematics but also a greater number of students who have the potential and interest to pursue advanced academic learning in mathematics. Because many jobs directly and indirectly require facility with different aspects of applied mathematics (Rivera-Batiz 1992), achieving the goals of this framework will also enable California students to pursue the broadest possible range of career choices.

By meeting the goals of standards-based mathematics, students will achieve greater proficiency in the practical uses of mathematics in everyday life, such as balancing a checkbook, purchasing a car, and understanding the daily news. This process will help the citizens of California understand their world and be productive members of society.

When students delve deeply into mathematics, they gain not only conceptual understanding of mathematical principles but also knowledge of and experience with pure reasoning. One of the most important goals of mathematics is to teach students logical reasoning. Mathematical reasoning and conceptual understanding are not separate from content; they are intrinsic to the mathematical discipline that students master at the more advanced levels.

Students who understand the aesthetics and power of mathematics will have a deep understanding of how mathematics enriches their lives. When students experience the satisfaction of mastering a challenging area of human thought, they feel better about themselves (Nicholls 1984). Students who can see the interdependence of mathematics and music, art, architecture, science, philosophy,

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and other disciplines will become lifelong students of mathematics regardless of the career they pursue.

When students master or exceed the goals of standards-based mathematics instruction, the benefits to both the individual and to society are enormous. Yet achieving these goals is no simple task. Hard work lies ahead. This framework was designed to help educators, families, and communities in California to meet the challenge.

Achieving Balance Within Mathematics— Three Important Components

At the heart of mathematics is reasoning. One cannot do mathematics without reasoning. . . . Teachers need to provide their students with many opportunities to reason through their solutions, conjectures, and thinking processes. Opportunities in which very young students . . . make distinctions between irrelevant and relevant information or attributes, and justify relationships between sets can contribute to their ability to reason logically.

—S. Chapin, *The Partners in Change Handbook*

Mathematics education must provide students with a balanced instructional program. In such a program students become proficient in basic computational and procedural skills, develop conceptual understanding, and become adept at problem solving.

All three components are important; none is to be neglected or under-emphasized. Balance, however, does not imply allocating set amounts of time for each of the three components. At some times students might be concentrating on lessons or tasks that focus on one component; at other times the focus may be on two or all three. As described in Chapter 4, “Instructional Strategies,” different types of instruction seem to foster different components of mathematical competence. Nonetheless, recent studies suggest that all three components are interrelated (Geary 1994; Siegler and Stern 1998; Sophian 1997). For example, conceptual understanding provides important constraints on the types of procedures children use to solve mathematics problems; at the same time practicing procedures provides an opportunity to make inductions about the underlying concepts (Siegler and Stern 1998).

Balance Defined

Computational and procedural skills are those that all students should learn to use routinely and automatically. Students should practice basic computational and procedural skills sufficiently and use them frequently enough to commit them to memory. Frequent use is also required to ensure that these skills are retained and maintained over the years.

Mathematics makes sense to students who have a conceptual understanding of the domain. They know not only how to apply skills but also when to apply them

and why they are being applied. They see the structure and logic of mathematics and use it flexibly, effectively, and appropriately. In seeing the larger picture and in understanding the underlying concepts, they are in a stronger position to apply their knowledge to new situations and problems and to recognize when they have made procedural errors.

Students who do not have a deep understanding of mathematics suspect that it is just a jumble of unrelated procedures and incomprehensible formulas. For example, children who do not understand the basic counting concepts view counting as a rote, a mechanical activity. They believe that the only correct way to count is by starting from left to right and by assigning each item a number (with a number name, such as “one”) in succession (Briars and Siegler 1984). In contrast, children with a good conceptual understanding of counting understand that items can be counted in any order—starting from right to left, skipping around, and so forth—as long as each item is counted only once (Gelman and Meck 1983). A strong conceptual understanding of counting, in turn, provides the foundation for using increasingly sophisticated counting strategies to solve arithmetic problems (Geary, Bow-Thomas, and Yao 1992).

Problem solving in mathematics is a goal-related activity that involves applying skills, understandings, and experiences to resolve new, challenging, or perplexing mathematical situations. Problem solving involves a sequence of activities directed toward a specific mathematical goal, such as solving a word problem, a task that often involves the use of a series of mathematical procedures and a conceptual representation of the problem to be solved (Geary 1994; Siegler and Crowley 1994; Mayer 1985).

When students apply basic computational and procedural skills and understandings to solve new or perplexing problems, their basic skills are strengthened, the challenging problems they encounter can become routine, and their conceptual understanding deepens. They come to see mathematics as a way of finding solutions to problems that occur outside the classroom. Thus, students grow in their ability and persistence in problem solving through experience in solving problems at a variety of levels of difficulty and at every level in their mathematical development.

Basic Computational and Procedural Skills

For each level of mathematics, a specific set of basic computational and procedural skills must be learned. For example, students need to memorize the number facts of addition and multiplication of one-digit numbers and their corresponding subtraction and division facts. The ability to retrieve these facts automatically from long-term memory, in turn, makes the solving of more complex problems, such as multistep problems that involve basic arithmetic, quicker and less likely to result in errors (Geary and Widaman 1992). As students progress through elementary school, middle school, and high school, they should become proficient in the following skills:

- Finding correct answers to addition, subtraction, multiplication, and division problems

The students come to see mathematics as a way of finding solutions to problems that occur outside the classroom.

- Finding equivalencies for fractions, decimals, and percents
- Performing operations with fractions, decimals, and percents
- Measuring
- Finding perimeters and areas of simple figures
- Interpreting graphs encountered in daily life
- Finding the mean and median of a set of data from the real world
- Using scientific notation to represent very large or very small numbers
- Using basic geometry, including the Pythagorean theorem
- Finding the equation of a line, given two points through which it passes
- Solving linear equations and systems of linear equations

This list, which is by no means exhaustive, is provided for illustrative purposes only. Several factors should be considered in the development and maintenance of basic computational and procedural skills:

- Students must practice skills in order to become proficient. Practice should be varied and should be included both in homework assignments and in classroom activities. Teachers, students, and parents should realize that students must spend substantial time and exert significant effort to learn a skill and to maintain it for the long term (Ericsson, Krampe, and Tesch-Römer 1993).
- Basic computational and procedural skills develop over time, and they increase in depth and complexity through the years. For example, the ability to interpret information presented graphically begins at the primary level and extends to more sophisticated procedures as students progress through the grades.
- The development of basic computational and procedural skills requires that students be able to distinguish among different basic procedures by understanding what the procedures do. Only then will students have the basis for determining when to use the procedures they learn. For example, students must know the procedures involved in adding and multiplying fractions, and they must understand how and why these procedures produce different results.
- To maintain skills, students must use them frequently. Once students have learned to use the Pythagorean theorem, for example, they need to use it again and again in algebraic and geometric problems.
- Students may sometimes learn a skill more readily when they know how it will be used or when they are intrigued by a problem that requires the skill.

Conceptual Understanding

Conceptual understanding is important at all levels of study. For example, during the elementary grades students should understand that:

- One way of thinking about multiplication is as repeated addition.
- One interpretation of fractions is as parts of a whole.
- Measurement of distances is fundamentally different from measurement of area.
- A larger sample generally provides more reliable information about the probability of an event than does a smaller sample.

As students progress through middle school and high school, they should, for example, understand that:

- The concepts of proportional relationships underlie similarity.
- The level sets of functions of two variables are curves in the coordinate plane.
- Factoring a polynomial function into irreducible factors helps locate the x -intercepts of its graph.
- Proofs are required to establish the truth of mathematical theorems.

Problem Solving

Problems occur in many forms. Some are simple and routine, providing practice for skill development. Others are more complex and take a longer time to complete. Whatever their nature, it is important that the kinds of problems students are asked to solve balance situations in the real world with more abstract situations. The process of solving problems generally has the following stages (Geary 1994; Mayer 1985):

- Formulation, analysis, and translation
- Integration and representation
- Solutions and justifications

Formulation, analysis, and translation. Problems may be stated in an imprecise form or in descriptions of puzzling or complex situations. The ability to recognize potential mathematical relationships is an important problem-solving technique, as is the identification of basic assumptions made directly or indirectly in the description of the situation, including the identification of extraneous or missing information. Important considerations in the formulation and analysis of any problem situation include determining mathematical hypotheses, making conjectures, recognizing existing patterns, searching for connections to known mathematical structures, and translating the gist of the problem into mathematical representations (e.g., equations).

Integration and representation. Important skills involved in the translation of a mathematical problem into a solvable equation are problems of integration and representation. Integration involves putting together different pieces of information that are presented in complex problems, such as multistep problems. However such problems are represented, a wide variety of basic and technical skills are needed in solving problems; and, given this need, a mathematics program should include a substantial number of ready-to-solve exercises that are designed specifically to develop and reinforce such skills.

Solutions and justifications. Students should have a range of strategies to use in solving problems and should be encouraged to think about all possible procedures that might be used to aid in the solving of any particular problem, including but not limited to the following:

- Referring to and developing graphs, tables, diagrams, and sketches
- Computing
- Finding a simpler related problem
- Looking for patterns
- Estimating, conjecturing, and verifying
- Working backwards

Once the information in a complex problem has been integrated and translated into a mathematical representation, the student must be skilled at solving the associated equations and verifying the correctness of the solutions. Students might also identify relevant mathematical generalizations and seek connections to similar problems. From the earliest years students should be able to communicate and justify their solutions, starting with informal mathematical reasoning and advancing over the years to more formal mathematical proofs.

Connecting Skills, Conceptual Understanding, and Problem Solving

Basic computational and procedural skills, conceptual understanding, and problem solving form a web of mutually reinforcing elements in the curriculum. Computational and procedural skills are necessary for the actual solution of both simple and complex problems, and the practice of these skills provides a context for learning about the associated concepts and for discovering more sophisticated ways of solving problems (Siegler and Stern 1998). The development of conceptual understanding provides necessary constraints on the types of procedures students use to solve mathematics problems, enables students to detect when they have committed a procedural error, and facilitates the representation and translation phases of problem solving. Similarly, the process of applying skills in varying and increasingly complex problem-solving situations is one of the ways in which students not only refine their skills but also reinforce and strengthen their conceptual understanding and procedural competencies.

Key Components of an Effective Mathematics Program

Assumption: Proficiency is determined by student performance on valid and reliable measures aligned with the mathematics standards.

In an effective and well-designed mathematics program, students move steadily from what they already know to a mastery of skills, knowledge, and understanding. Their thinking progresses from an ability to explain what they are doing, to an ability to justify how and why they are doing it, to a stage at which they can derive formal proofs. The quality of instruction is a key factor in developing student proficiency in mathematics. In addition, several other factors or program components play an important role. They are discussed in the following section:

I. Assessment

Assessment should be the basis for instruction, and different types of assessment interact with the other components of an effective mathematics program.

In an effective mathematics program:

- Assessment is aligned with and guides instruction. Students are assessed frequently to determine whether they are progressing steadily toward achieving the standards, and the results of this assessment are useful in determining instructional priorities and modifying curriculum and instruction. The assessment looks at the same balance (computational and procedural skills, conceptual understanding, and problem solving) emphasized in instruction.
- Assessments serve different purposes and are designed accordingly. Assessment for determining a student's placement in a mathematics program should cover a broad range of standards. These broad assessments measure whether or not students have prerequisite knowledge and allow them to demonstrate their full understanding of mathematics. Monitoring student progress daily or weekly requires a quick and focused measurement tool. Summative evaluation, which takes place at the end of a series of lessons or a course, provides specific and detailed information about which standards have or have not been achieved.
- Assessments are valid and reliable. A valid assessment measures the specific content it was designed to measure. An assessment instrument is reliable if it is relatively error-free and provides a stable result each time it is administered.
- Assessment can improve instruction when teachers use the results to analyze what students have learned and to identify difficult concepts that need reteaching.
- Assessments for specially designed instructional materials for students having difficulty achieving at grade level (mathematics intervention and algebra readiness program materials) must be extensive and should determine a student's need and placement in the program, existing strengths and weaknesses, and progress and mastery of the materials.

II. Instruction

The quality of instruction is the single most important component of an effective mathematics program. International comparisons show a high correlation between the quality of mathematics instruction and student achievement (Beaton et al. 1996).

In an effective mathematics program:

- Teachers possess an in-depth understanding of the content standards and the mathematics they are expected to teach and continually strive to increase their knowledge of content.
- Teachers are able to select research-based instructional strategies that are appropriate to the instructional goals and to students' needs.
- Teachers effectively organize instruction around goals that are tied to the standards and direct students' mathematical learning.
- Teachers use the results of assessment to guide instruction.

III. Instructional Time

Study after study has demonstrated the relationship between the time on task and student achievement (Stigler, Lee, and Stevenson 1987, 1283). Priority

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must be given to the teaching of mathematics, and instructional time must be protected from interruptions.

In an effective mathematics program:

- Adequate time is allocated to mathematics. Every day all students receive at least 50 to 60 minutes of mathematics instruction, not including homework. **Additional instructional time is allocated for students who are, for whatever reason, performing substantially below grade level in mathematics.** All students are encouraged to take mathematics courses throughout high school.
- Learning time is extended through homework that increases in complexity and duration as students mature. Homework should be valued and reviewed. The purpose of homework is to practice skills previously taught or to have students apply their previously learned knowledge and skills to new problems. It should be assigned in amounts that are grade-level appropriate and, at least in the early grades, it should focus on independent practice and the application of skills already taught. For more advanced students, homework may be used as a means for exploring new concepts.
- During the great majority of allocated time, students are active participants in the instruction. *Active* can be described as the time during which students are engaged in thinking about mathematics or doing mathematics.
- Instructional time for mathematics is maximized and protected from such interruptions as calls to the office, public address announcements, and extra-curricular activities.

IV. Instructional Resources

All teachers need high-quality instructional resources, but new teachers especially depend on well-designed resources and materials that are aligned with the standards.

In an effective mathematics program:

- Instructional resources focus on the grade-level standards. It may be necessary to go beyond the standards, however, both to provide meaningful enrichment and acceleration for fast learners and to introduce content needed for the mastery of standards in subsequent grades and courses. For example, the Algebra I standards do not mention complex numbers; yet quadratic equations, which often have complex roots, are fully developed in Algebra I. Therefore, an introduction to complex numbers may be included in Algebra I, both to avoid the artificial constraint of having only problems with real roots and to lay the foundation for the mastery of complex numbers in Algebra II.
- Instructional resources are factually and technically accurate and address the content outlined in the standards.
- Instructional resources emphasize depth of coverage. The most critical, highest-priority standards are addressed in the greatest depth. Ample practice is provided.
- Instructional resources are organized in a sequential, logical way. The resources are coordinated from level to level.

- Instructional options for teachers are included. For instance, a teacher's guide might explain the rationale and procedures for different ways of introducing a topic (e.g., through direct instruction or discovery-oriented instruction) and present various methods for assessing student progress. In addition to providing teachers with options, the resources should offer reliable guidelines for exercising those options.
- Resources balance basic computational and procedural skills, conceptual understanding, and problem solving and stress the interdependency of all three.
- Resources provide ample opportunities for students to explain their thinking, verbally and in writing, formally and informally.
- Resources supply ideas or tools for accommodating diverse student performance within any given classroom. They offer suggestions for reteaching a concept, providing additional practice for struggling students, or condensing instruction so that advanced students can concentrate on new material.

V. Instructional Grouping and Scheduling

Research shows that what students are taught has a greater effect on achievement than does how they are grouped (Kulik 1992; Rogers 1991). The first focus of educators should always be on the quality of instruction. Grouping and scheduling are tools that educators can use to improve learning, not goals in and of themselves.

In an effective mathematics program:

- Grouping students according to their instructional needs improves student achievement (Benbow and Stanley 1996). An effective mathematics program (1) uses grouping options in accordance with variability within individual classrooms; and (2) maintains or changes grouping strategies in accordance with student performance on regular assessments.
- Cooperative group work is used judiciously, supplementing and expanding on initial instruction either delivered by teachers or facilitated through supervised exploration. Although students can often learn a great deal from one another and can benefit from the opportunity to discuss their thinking, the teacher is the primary leader in a class and maintains an active instructional role during cooperative learning. When cooperative group work is used, it should lead toward students' eventual independent demonstration of mastery of the standards and individual responsibility for learning.
- Cross-grade or cross-class grouping is an alternative to the more arbitrary practice of grouping according to chronological age or grade. Grouping by instructional needs across grade levels increases scheduling challenges for teachers and administrators near the beginning of a school year, but many teachers find the practice liberating later on because it reduces the number of levels for which a teacher must be prepared to teach in a single period.

VI. Classroom Management

Potentially, the primary management tool for teachers is the mathematics curriculum itself. When students are actively engaged in focused, rigorous mathematics, fewer opportunities for inappropriate behavior arise. When students are successful and their successes are made clear to them, they are more likely to become motivated to work on mathematics.

In an effective mathematics program:

- Teachers are positive and optimistic about the prospect that all students can achieve. Research shows that teachers' self-esteem and enthusiasm for the subject matter have a greater effect on student achievement than does students' self-esteem (Clark 1997).
- Classrooms have a strong sense of purpose. Both academic and social expectations are clearly understood by teachers and students alike. Academic expectations relate directly to the standards.
- Intrinsic motivation is fostered by helping students to develop a deep understanding of mathematics, encouraging them to expend the effort needed to learn, and organizing instruction so that students experience satisfaction when they have mastered a difficult concept or skill. External reward systems are used sparingly; for example, as a temporary motivational device for older students who enter mathematics instruction without the intrinsic motivation to work hard.

VII. Professional Development

The preparation of teachers and support for their continuing professional development are critical to the quality of California schools. Research from other countries suggests that student achievement can improve when teachers are able to spend time together planning and evaluating instruction (Beaton et al. 1996).

In an effective mathematics program:

- Teachers have received excellent preservice training, are knowledgeable about mathematics content, and are able to use a wide variety of instructional strategies.
- Continuing teacher in-service training focuses on (1) enhancing teachers' proficiency in mathematics; (2) providing pedagogical tools that help teachers to ensure that all students meet or exceed grade-level standards; and (3) helping teachers understand the research behind the content and instructional design of their adopted instructional materials aligned with the standards and ways in which to make effective use of those materials.
- Staff development is a long-term, planned investment strongly supported by the administration and designed to ensure that teachers continue to develop skills and knowledge in mathematics content and instructional options. "One-shot" staff development activities with no relationship to a long-term plan are recognized as having little lasting value.
- As with students, staff development actively engages teachers in mathematics and mathematics instruction. In addition to active involvement during

classroom-style staff development, teachers have the opportunity to interact with students and staff developers during in-class coaching sessions.

- Individuals who have helped teachers bring their students to high achievement levels in mathematics are called on to demonstrate effective instructional practices with students.
- Teachers are given time and opportunities to work together to plan mathematics instruction. Districts and schools find creative ways to allow time for this planning.

VIII. Administrative Practices

Administrative support for mathematics instruction can help remind all those involved in education that reform efforts are not effective unless they contribute to increased achievement. Administrators can help teachers maintain a focus on high-quality instruction.

In an effective mathematics program:

- Mathematics achievement is among the highest priorities at the school.
- Long-term and short-term goals for the school, each grade level, and individuals are outlined clearly and reviewed frequently.
- Scheduling, grouping, and allocating personnel are shaped by a determination that all students will meet or exceed the mathematics standards.
- Principals demonstrate a strong sense of personal responsibility for achievement within their schools.
- Administrators consider using mathematics specialists to teach most or all of the mathematics classes or to coach other teachers.
- Administrators plan in advance for predictable contingencies, such as the need to realign instructional groups frequently, accommodate students transferring into the school, or redesign instruction to include intervention for students performing below grade level.
- Administrators and teachers collaborate on developing schoolwide management systems and schoolwide efforts to showcase mathematics for students, parents, and other members of the community.

IX. Community Involvement

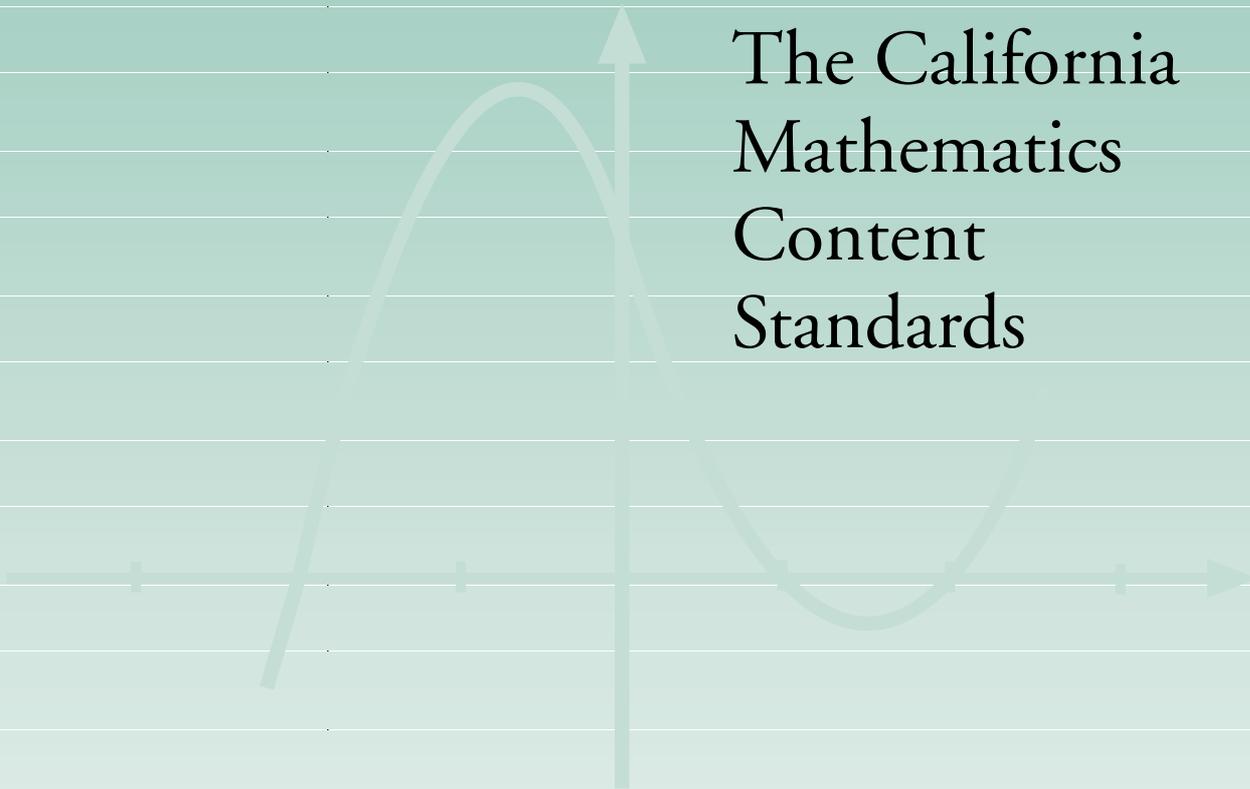
Mathematics education is everybody's business. Parents, community members, and business and industry can all make significant contributions.

In an effective mathematics program:

- Parents are encouraged to be involved in education and are assisted in supporting their children's learning in mathematics. Parents' comments are encouraged, valued, and used for program planning.
- Materials are organized so that parents, siblings, and community members can provide extended learning experiences.
- The community is used as a classroom that offers abundant examples of how and why mathematics is important in people's lives, work, and thinking.

$$2 \frac{150}{100} > 135$$

The California
Mathematics
Content
Standards



$$f(x) = \frac{2}{x}$$

A high-quality mathematics program is essential for all students and provides every student with the opportunity to choose among the full range of future career paths. Mathematics, when taught well, is a subject of beauty and elegance, exciting in its logic and coherence. It trains the mind to be analytic—providing the foundation for intelligent and precise thinking.

To compete successfully in the worldwide economy, today's students must have a high degree of comprehension in mathematics. For too long schools have suffered from the notion that success in mathematics is the province of a talented few. Instead, a new expectation is needed: all students will attain California's mathematics academic content standards, and many will be inspired to achieve far beyond the minimum standards.

These content standards establish what every student in California can and needs to learn in mathematics. They are comparable to the standards of the most academically demanding nations, including Japan and Singapore—two high-performing countries in the Third International Mathematics and Science Study (TIMSS). Mathematics is critical for all students, not only those who will have careers that demand advanced mathematical preparation but all citizens who will be living in the twenty-first century. These standards are based on the premise that all students are capable of learning rigorous mathematics and learning it well, and all are capable of learning far more than is currently expected. Proficiency in most of mathematics is not an innate characteristic; it is achieved through persistence, effort, and practice on the part of students and rigorous and effective instruction on the part of teachers. Parents and teachers must provide support and encouragement.

The standards focus on essential content for all students and prepare students for the study of advanced mathematics, science and technical careers, and post-secondary study in all content areas. All students are required to grapple with solving problems; develop abstract, analytic thinking skills; learn to deal effectively and comfortably with variables and equations; and use mathematical notation effectively to model situations. The goal in mathematics education is for students to:

- Develop fluency in basic computational skills.
- Develop an understanding of mathematical concepts.
- Become mathematical problem solvers who can recognize and solve routine problems readily and can find ways to reach a solution or goal where no routine path is apparent.
- Communicate precisely about quantities, logical relationships, and unknown values through the use of signs, symbols, models, graphs, and mathematical terms.
- Reason mathematically by gathering data, analyzing evidence, and building arguments to support or refute hypotheses.
- Make connections among mathematical ideas and between mathematics and other disciplines.

These content standards establish what every student in California can and needs to learn in mathematics.

The standards identify what all students in California public schools should know and be able to do at each grade level. Nevertheless, local flexibility is maintained with these standards. Topics may be introduced and taught at one or two grade levels before mastery is expected. Decisions about how best to teach the standards and in what order they should be taught are left to teachers, schools, and school districts.

The standards emphasize computational and procedural skills, conceptual understanding, and problem solving. These three components of mathematics instruction and learning are not separate from each other; instead, they are intertwined and mutually reinforcing.

Basic, or computational and procedural, skills are those skills that all students should learn to use routinely and automatically. Students should practice basic skills sufficiently and frequently enough to commit them to memory.

Mathematics makes sense to students who have a conceptual understanding of the domain. They know not only *how* to apply skills but also *when* to apply them and *why* they should apply them. They understand the structure and logic of mathematics and use the concepts flexibly, effectively, and appropriately. In seeing the big picture and in understanding the concepts, they are in a stronger position to apply their knowledge to situations and problems they may not have encountered before and readily recognize when they have made procedural errors.

The mathematical reasoning standards are different from the other standards in that they do not represent a content domain. Mathematical reasoning is involved in all strands.

The standards do not specify how the curriculum should be delivered. Teachers may use direct instruction, explicit teaching, or knowledge-based discovery learning; investigatory, inquiry-based, problem-solving-based, guided discovery, set-theory-based, traditional, or progressive methods; or other ways in which to teach students the subject matter set forth in these standards. At the middle and high school levels, schools can use the standards with an integrated program or with the traditional course sequence of Algebra I, geometry, Algebra II, and so forth.

Schools that use these standards “enroll” students in a mathematical apprenticeship in which they practice skills, solve problems, apply mathematics to the real world, develop a capacity for abstract thinking, and ask and answer questions involving numbers or equations. Students need to know basic formulas, understand what they mean and why they work, and know when they should be applied. Students are also expected to struggle with thorny problems after learning to perform the simpler calculations on which they are based.

Teachers should guide students to think about why mathematics works in addition to how it works and should emphasize understanding of mathematical concepts as well as achievement of mathematical results. Students need to recognize that the solution to any given problem may be determined by employing more than one strategy and that the solution frequently raises new questions of its own: Does the answer make sense? Are there other, more efficient ways to arrive

at the answer? Does the answer bring up more questions? Can I answer those? What other information do I need?

Problem solving involves applying skills, understanding, and experiences to resolve new or perplexing situations. It challenges students to apply their understanding of mathematical concepts in a new or complex situation, to exercise their computational and procedural skills, and to see mathematics as a way of finding answers to some of the problems that occur outside a classroom. Students grow in their ability and persistence in problem solving by extensive experience in solving problems at a variety of levels of difficulty and at every level in their mathematical development.

Problem solving, therefore, is an essential part of mathematics and is subsumed in every strand and in each of the disciplines in grades eight through twelve. Problem solving is not separate from content. Rather, students learn concepts and skills in order to apply them to solve problems in and outside school. Because problem solving is distinct from a content domain, its elements are consistent across grade levels.

The problems that students solve must address important mathematics. As students progress from grade to grade, they should deal with problems that (1) require increasingly more advanced knowledge and understanding of mathematics; (2) are increasingly complex (applications and purely mathematical investigations); and (3) require increased use of inductive and deductive reasoning and proof. In addition, problems should increasingly require students to make connections among mathematical ideas within a discipline and across domains. Each year students need to solve problems from all strands, although most of the problems should relate to the mathematics that students study that year. A good problem is one that is mathematically important; specifies the problem to be solved but not the solution path; and draws on grade-level appropriate skills and conceptual understanding.

Organization of the Standards

The mathematics content standards for kindergarten through grade seven are organized by grade level and are presented in five strands: Number Sense; Algebra and Functions; Measurement and Geometry; Statistics, Data Analysis, and Probability; and Mathematical Reasoning. Focus statements indicating the increasingly complex mathematical skills that will be required of students from kindergarten through grade seven are included at the beginning of each grade level; the statements indicate the ways in which the discrete skills and concepts form a cohesive whole. [The symbol ● identifies the key standards to be covered in kindergarten through grade seven.]

The standards for grades eight through twelve are organized differently from those for kindergarten through grade seven. Strands are not used for organizational purposes because the mathematics studied in grades eight through twelve falls naturally under the discipline headings algebra, geometry, and so forth. Many schools teach this material in traditional courses; others teach it in an

Technology does not replace the need for all students to learn and master basic mathematics skills.

integrated program. To allow local educational agencies and teachers flexibility, the standards for grades eight through twelve do not mandate that a particular discipline be initiated and completed in a single grade. The content of these disciplines must be covered, and students enrolled in these disciplines are expected to achieve the standards regardless of the sequence of the disciplines.

Mathematics Standards and Technology

As rigorous mathematics standards are implemented for all students, the appropriate role of technology in the standards must be clearly understood.

The following considerations may be used by schools and teachers to guide their decisions regarding mathematics and technology:

Students require a strong foundation in basic skills. Technology does not replace the need for all students to learn and master basic mathematics skills. All students must be able to add, subtract, multiply, and divide easily without the use of calculators or other electronic tools. In addition, all students need direct work and practice with the concepts and skills underlying the rigorous content described in the *Mathematics Content Standards for California Public Schools* so that they develop an understanding of quantitative concepts and relationships. The students' use of technology must build on these skills and understandings; it is not a substitute for them.

Technology should be used to promote mathematics learning. Technology can help promote students' understanding of mathematical concepts, quantitative reasoning, and achievement when used as a tool for solving problems, testing conjectures, accessing data, and verifying solutions. When students use electronic tools, databases, programming language, and simulations, they have opportunities to extend their comprehension, reasoning, and problem-solving skills beyond what is possible with traditional print resources. For example, graphing calculators allow students to see instantly the graphs of complex functions and to explore the impact of changes. Computer-based geometry construction tools allow students to see figures in three-dimensional space and experiment with the effects of transformations. Spreadsheet programs and databases allow students to key in data and produce various graphs as well as compile statistics. Students can determine the most appropriate ways to display data and quickly and easily make and test conjectures about the impact of change on the data set. In addition, students can exchange ideas and test hypotheses with a far wider audience through the Internet. Technology may also be used to reinforce basic skills through computer-assisted instruction, tutoring systems, and drill-and-practice software.

The focus must be on mathematics content. The focus must be on learning mathematics, using technology as a tool rather than as an end in itself. Technology makes more mathematics accessible and allows one to solve mathematical problems with speed and efficiency. However, technological tools cannot be used effectively without an understanding of mathematical skills, concepts, and relationships. As students learn to use electronic tools, they must also develop the quantitative reasoning necessary to make full use of those tools. They must also

have opportunities to reinforce their estimation and mental math skills and the concept of place value so that they can quickly check their calculations for reasonableness and accuracy.

Technology is a powerful tool in mathematics. When used appropriately, technology may help students develop the skills, knowledge, and insight necessary to meet rigorous content standards in mathematics and make a successful transition to the world beyond school. The challenge for educators, parents, and policymakers is to ensure that technology supports, but is not a substitute for, the development of quantitative reasoning and problem-solving skills.¹

¹ Complete citations for the sources following some of the mathematics problems in this chapter appear in “Works Cited” at the end of this publication.

Kindergarten Mathematics Content Standards

By the end of kindergarten, students understand small numbers, quantities, and simple shapes in their everyday environment. They count, compare, describe and sort objects and develop a sense of properties and patterns.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified, particularly in the primary grades, before they are used with students.

The symbol ● identifies the key standards for kindergarten.

Number Sense

1.0 Students understand the relationship between numbers and quantities (i.e., that a set of objects has the same number of objects in different situations regardless of its position or arrangement):

- 1.1 Compare two or more sets of objects (up to 10 objects in each group) and identify which set is equal to, more than, or less than the other.

Are there more circles or more triangles in the following collection?



- 1.2 Count, recognize, represent, name, and order a number of objects (up to 30).

Which numbers are missing if we are counting by ones?

11, 12, 13, __, __, 16, 17, __, __, __, 21, 22, 23, 24.

- 1.3 Know that the larger numbers describe sets with more objects in them than the smaller numbers have.

2.0 Students understand and describe simple additions and subtractions:

- 2.1 Use concrete objects to determine the answers to addition and subtraction problems (for two numbers that are each less than 10).

Pair up as many groups of beans from the left column with groups of beans from the right column so that each group adds up to 10 beans.

3.0 Students use estimation strategies in computation and problem solving that involve numbers that use the ones and tens places:

- 3.1 Recognize when an estimate is reasonable.

Algebra and Functions

1.0 Students sort and classify objects:

- 1.1** Identify, sort, and classify objects by attribute and identify objects that do not belong to a particular group (e.g., all these balls are green, those are red).

Students compare objects:

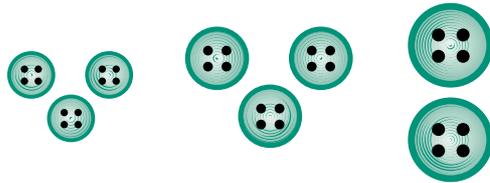
1. Which pencil is longer? Shorter?



2. Describe how the following 2 objects are the same or different.



3. Show students buttons sorted into 3 sets as shown and ask them to identify how buttons were sorted.



Kindergarten

Measurement and Geometry

- 1.0** Students understand the concept of time and units to measure it; they understand that objects have properties, such as length, weight, and capacity, and that comparisons may be made by referring to those properties:

- 1.1 Compare the length, weight, and capacity of objects by making direct comparisons with reference objects (e.g., note which object is shorter, longer, taller, lighter, heavier, or holds more).

Who is the tallest girl in the class? The tallest boy? Which container holds more?

- 1.2 Demonstrate an understanding of concepts of time (e.g., morning, afternoon, evening, today, yesterday, tomorrow, week, year) and tools that measure time (e.g., clock, calendar).

If the teacher says to a class that a substitute will be teaching for the next four school days, when can the class expect their teacher to probably return? Tomorrow? Next week? Next month? Next year?

- 1.3 Name the days of the week.

- 1.4 Identify the time (to the nearest hour) of everyday events (e.g., lunch time is 12 o'clock; bedtime is 8 o'clock at night).

2.0 Students identify common objects in their environment and describe the geometric features:

- 2.1 Identify and describe common geometric objects (e.g., circle, triangle, square, rectangle, cube, sphere, cone).

Which of these is a square?



Given 5 squares of the same size, can you make use of some or all of them to form a bigger square?



- 2.2 Compare familiar plane and solid objects by common attributes (e.g., position, shape, size, roundness, number of corners).

Kindergarten

Statistics, Data Analysis, and Probability

1.0 Students collect information about objects and events in their environment:

- 1.1 Pose information questions; collect data; and record the results using objects, pictures, and picture graphs.
- 1.2** Identify, describe, and extend simple patterns (such as circles or triangles) by referring to their shapes, sizes, or colors.

Mathematical Reasoning

1.0 Students make decisions about how to set up a problem:

- 1.1 Determine the approach, materials, and strategies to be used.
- 1.2 Use tools and strategies, such as manipulatives or sketches, to model problems.

2.0 Students solve problems in reasonable ways and justify their reasoning:

- 2.1 Explain the reasoning used with concrete objects and/or pictorial representations.
- 2.2 Make precise calculations and check the validity of the results in the context of the problem.

In a bag there are 4 apples, 3 oranges, 5 bananas, and 3 water bottles. How many pieces of fruit are in the bag altogether? How many different kinds of fruit are in the bag? How many objects altogether are in the bag?

Grade One Mathematics Content Standards

By the end of grade one, students understand and use the concept of ones and tens in the place value number system. Students add and subtract small numbers with ease. They measure with simple units and locate objects in space. They describe data and analyze and solve simple problems.

Number Sense

1.0 Students understand and use numbers up to 100:

- 1.1 Count, read, and write whole numbers to 100.
- 1.2 Compare and order whole numbers to 100 by using the symbols for less than, equal to, or greater than ($<$, $=$, $>$).

Which of the following are correct and which are incorrect?

(a) $75 > 76$ (b) $48 < 42$ (c) $89 > 91$ (d) $59 < 67$ (e) $34 = 33$

- 1.3 Represent equivalent forms of the same number through the use of physical models, diagrams, and number expressions (to 20) (e.g., 8 may be represented as $4 + 4$, $5 + 3$, $2 + 2 + 2 + 2$, $10 - 2$, $11 - 3$).
- 1.4 Count and group objects in ones and tens (e.g., three groups of 10 and 4 equals 34, or $30 + 4$).

A certain brand of chewing gum has 10 pieces in each pack. If there are 14 students, what is the smallest number of packs we must buy to make sure each student gets at least one piece of gum? If there are 19 students? What about 21 students?

There are 5 quarters, 9 dimes, 3 nickels, and 8 pennies. They are supposed to be put in piles of ten (coins). How many such piles can be formed by all these coins, and how many are left over?

- 1.5 Identify and know the value of coins and show different combinations of coins that equal the same value.

Give each student a plastic set of 25 pennies, 5 nickels, and 2 dimes. Ask the class to find different ways to make 25 cents.

2.0 Students demonstrate the meaning of addition and subtraction and use these operations to solve problems:

- 2.1 Know the addition facts (sums to 20) and the corresponding subtraction facts and commit them to memory.
I had 10 cupcakes, but I ate 3 of them. How many cupcakes do I have left? How many if I had 18 and ate 5?
- 2.2 Use the inverse relationship between addition and subtraction to solve problems.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified, particularly in the primary grades, before they are used with students.

The symbol ● identifies the key standards for grade one.

2.3 Identify one more than, one less than, 10 more than, and 10 less than a given number.

2.4 Count by 2s, 5s, and 10s to 100.

Which numbers are missing if we are counting by 2s?

24, 26, 28, 30, __, __, 36, __, 40, 42, 44, __, __, 50

Which numbers are missing if we are counting by 5s?

15, 20, 25, 30, __, __, 45, __, 55, 60, __, 70, __, 80

2.5 Show the meaning of addition (putting together, increasing) and subtraction (taking away, comparing, finding the difference).

2.6 Solve addition and subtraction problems with one- and two-digit numbers (e.g., $5 + 58 = \underline{\quad}$).

If I read 16 pages on Monday, 9 pages on Tuesday, no pages on Wednesday, and 7 pages on Thursday, how many pages have I read so far this week?

2.7 Find the sum of three one-digit numbers.

3.0 Students use estimation strategies in computation and problem solving that involve numbers that use the ones, tens, and hundreds places:

3.1 Make reasonable estimates when comparing larger or smaller numbers.

Algebra and Functions

1.0 Students use number sentences with operational symbols and expressions to solve problems:

1.1 Write and solve number sentences from problem situations that express relationships involving addition and subtraction.

Do the following problems in succession:

Take away

Marie had some pencils in her desk. She put 5 more in her desk. Then she had 14. How many pencils did she have in her desk to start with?

Comparison

Eddie had 14 helium balloons. A number of them floated away. He had 5 left. How many did he lose?

Difference

1. Nina had 14 seashells. That was 5 more than Pedro had. How many seashells did Pedro have?

2. $5 + \underline{\quad} = 6$? $\underline{\quad} + 12 = 14$?

- 1.2 Understand the meaning of the symbols $+$, $-$, $=$.
- 1.3 Create problem situations that might lead to given number sentences involving addition and subtraction.

Measurement and Geometry

1.0 Students use direct comparison and nonstandard units to describe the measurements of objects:

Grade One

- 1.1 Compare the length, weight, and volume of two or more objects by using direct comparison or a nonstandard unit.

Measure your desk by using the length of a ballpoint pen. How many ballpoint pens would be roughly equal to the length of your desk? The width of your desk? Which is longer?

- 1.2 Tell time to the nearest half hour and relate time to events (e.g., before/after, shorter/longer).

2.0 Students identify common geometric figures, classify them by common attributes, and describe their relative position or their location in space:

- 2.1 Identify, describe, and compare triangles, rectangles, squares, and circles, including the faces of three-dimensional objects.

Describe the shape of a page in your textbook and compare it to the face of the clock on the wall.

- 2.2 Classify familiar plane and solid objects by common attributes, such as color, position, shape, size, roundness, or number of corners, and explain which attributes are being used for classification.

- 2.3 Give and follow directions about location.

Here are pictures on a table of a ball, a girl, a horse, and a cat. Arrange them according to these directions:

1. Put the picture of the ball above the picture of the horse.
2. Put the picture of the girl on top of the picture of the horse.
3. Put the picture of the cat under the picture of the horse.

- 2.4 Arrange and describe objects in space by proximity, position, and direction (e.g., near, far, below, above, up, down, behind, in front of, next to, left or right of).

Statistics, Data Analysis, and Probability

1.0 Students organize, represent, and compare data by category on simple graphs and charts:

- 1.1 Sort objects and data by common attributes and describe the categories.
- 1.2 Represent and compare data (e.g., largest, smallest, most often, least often) by using pictures, bar graphs, tally charts, and picture graphs.

2.0 Students sort objects and create and describe patterns by numbers, shapes, sizes, rhythms, or colors:

- 2.1** Describe, extend, and explain ways to get to a next element in simple repeating patterns (e.g., rhythmic, numeric, color, and shape).

Mathematical Reasoning

1.0 Students make decisions about how to set up a problem:

- 1.1 Determine the approach, materials, and strategies to be used.
- 1.2 Use tools, such as manipulatives or sketches, to model problems.

2.0 Students solve problems and justify their reasoning:

- 2.1 Explain the reasoning used and justify the procedures selected.
- 2.2 Make precise calculations and check the validity of the results from the context of the problem.

3.0 Students note connections between one problem and another.

Grade Two Mathematics Content Standards

By the end of grade two, students understand place value and number relationships in addition and subtraction, and they use simple concepts of multiplication. They measure quantities with appropriate units. They classify shapes and see relationships among them by paying attention to their geometric attributes. They collect and analyze data and verify the answers.

Number Sense

1.0 Students understand the relationship between numbers, quantities, and place value in whole numbers up to 1,000:

- 1.1** Count, read, and write whole numbers to 1,000 and identify the place value for each digit.
- 1.2 Use words, models, and expanded forms (e.g., $45 = 4 \text{ tens} + 5$) to represent numbers (to 1,000).

Kelly has 308 stickers. How many sets of hundreds, tens, and ones does she have?

- 1.3** Order and compare whole numbers to 1,000 by using the symbols $<$, $=$, $>$.

Which number sentence is true? (CST released test question, 2004)²

- (a) $359 < 375$ (b) $359 > 375$ (c) $359 < 359$ (d) $359 > 359$

2.0 Students estimate, calculate, and solve problems involving addition and subtraction of two- and three-digit numbers:

- 2.1** Understand and use the inverse relationship between addition and subtraction (e.g., an opposite number sentence for $8 + 6 = 14$ is $14 - 6 = 8$) to solve problems and check solutions.

Sophie did this subtraction problem. Which addition problem shows that she got the right answer? (CST released test question, 2004)

$$\begin{array}{r} 85 \\ -44 \\ \hline 41 \end{array}$$

- | | | | |
|--|--|--|--|
| $\begin{array}{r} 41 \\ +85 \\ \hline \end{array}$ | $\begin{array}{r} 44 \\ +85 \\ \hline \end{array}$ | $\begin{array}{r} 41 \\ +44 \\ \hline \end{array}$ | $\begin{array}{r} 44 \\ +44 \\ \hline \end{array}$ |
| (a) | (b) | (c) | (d) |

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified, particularly in the primary grades, before they are used with students.

The symbol ● identifies the key standards for grade two.

² The Web site for accessing the *California Standards Test (CST)* released test questions for mathematics is <http://cde.ca.gov/ta/tg/sr/css05rtq.asp>.

- 2.2** Find the sum or difference of two whole numbers up to three digits long.

Use drawings of tens and ones to help find the sum $37 + 17$ and the difference $25 - 19$. Now do the same problems again using addition and subtraction algorithms:

$$\begin{array}{r} 343 \\ + 265 \\ \hline \end{array} \quad \begin{array}{r} 748 \\ - 426 \\ \hline \end{array} \quad \begin{array}{r} 457 \\ + 324 \\ \hline \end{array}$$

Grade Two

Is $37 + 118$ the same as $100 + 30 + 10 + 7 + 8$?

- 2.3** Use mental arithmetic to find the sum or difference of two two-digit numbers.

In a game, Mysong and Naoki are making addition problems. They make two 2-digit numbers out of the four given numbers 1, 2, 3, and 4. Each number is used exactly once. The winner is the one who makes two numbers whose sum is the largest. Mysong had 43 and 21, while Naoki had 31 and 24. Who won the game? How do you know? Show how you can beat both Mysong and Naoki by making up two numbers with a larger sum than either. (Adapted from TIMSS, gr. 3–4, V-4a)³ (This problem also supports Mathematical Reasoning Standard 1.0.)

- 3.0** Students model and solve simple problems involving multiplication and division:

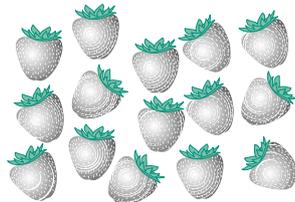
- 3.1** Use repeated addition, arrays, and counting by multiples to do multiplication.

Write 5×17 as a sum of numbers.

Draw a simple picture of seating 30 people in rows of 10. Show and explain how this is related to multiplication. Do this also for rows of 3, and again for rows of 5.

- 3.2** Use repeated subtraction, equal sharing, and forming equal groups with remainders to do division.

Kayla has these strawberries. She will give four strawberries to each of her three friends. How many strawberries will be left for Kayla? (CST released test question, 2004)



³ The "Web Resources" section in "Works Cited" shows the sources in which all mathematics problems from the Third International Mathematics and Science Study (TIMSS) appearing in this publication may be found. Each problem reproduced from TIMSS is copyrighted © 1994 by IEA, The Hague.

- 3.3** Know the multiplication tables of 2s, 5s, and 10s (to “times 10”) and commit them to memory.

There are nine benches in a park. There are two people sitting on each bench. How many people are sitting on the nine benches all together?
(CST released test question, 2004)



Grade Two

4.0 Students understand that fractions and decimals may refer to parts of a set and parts of a whole:

- 4.1** Recognize, name, and compare unit fractions from $\frac{1}{12}$ to $\frac{1}{2}$.

True or false?

- One-fourth of a pie is larger than one-sixth of the same pie.
- $\frac{1}{4} > \frac{1}{3}$
- $\frac{1}{8} < \frac{1}{10}$

- 4.2** Recognize fractions of a whole and parts of a group (e.g., one-fourth of a pie, two-thirds of 15 balls).

What fraction of this shape is shaded?
(CST released test question, 2004)



- 4.3** Know that when all fractional parts are included, such as four-fourths, the result is equal to the whole and to one.

Which fraction is equal to one whole?
(CST released test question, 2004)

- (a) $\frac{1}{3}$ (b) $\frac{1}{8}$ (c) $\frac{2}{3}$ (d) $\frac{8}{8}$

If you know that $379 + 363 = 742$, what is the sum of $363 + 379$?

What number goes in the box to make this number sentence true?
(CST released test question, 2004)

$$15 + 8 = \square + 15$$

- 1.2 Relate problem situations to number sentences involving addition and subtraction.

Andrew had 15 pennies. He found some more. Now he has 33. Which number sentence could be used to find how many pennies he found?
(CST released test question, 2004)

- (a) $15 + \underline{\quad} = 33$ (c) $\underline{\quad} - 33 = 15$
(b) $15 + 33 =$ (d) $\underline{\quad} - 15 = 33$

- 1.3 Solve addition and subtraction problems by using data from simple charts, picture graphs, and number sentences.

Grade Two

Measurement and Geometry

- 1.0 Students understand that measurement is accomplished by identifying a unit of measure, iterating (repeating) that unit, and comparing it to the item to be measured:**

- 1.1 Measure the length of objects by iterating (repeating) a nonstandard or standard unit.
- 1.2 Use different units to measure the same object and predict whether the measure will be greater or smaller when a different unit is used.

Four children measured the width of a room by counting how many paces it took them to cross it. It took Ana 9 paces, Erlane 8, Stephen 10, and Carlos 7. Who had the longest pace? (Adapted from TIMSS, gr. 3–4, L-8)

Measure the length of your desk with a new crayon and with a new pencil. Which is greater, the number of crayon units or the number of pencil units?

- 1.3** Measure the length of an object to the nearest inch and/or centimeter.
- 1.4 Tell time to the nearest quarter hour and know relationships of time (e.g., minutes in an hour, days in a month, weeks in a year).

Sean is going on vacation to visit his grandparents. He will be gone one month. About how many days will Sean be gone? (CST released test question, 2004)

- (a) 7 days (b) 30 days (c) 52 days (d) 365 days

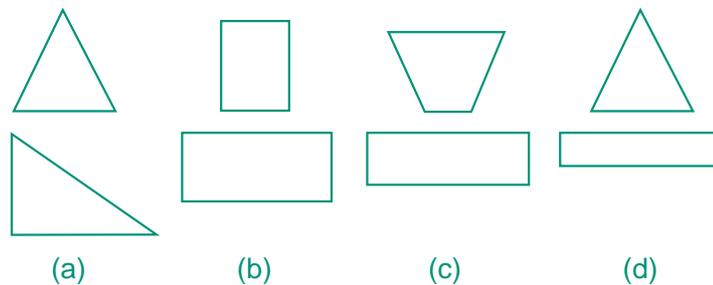
Which is a longer period: 3 weeks or 19 days? 27 days or 4 weeks?

- 1.5 Determine the duration of intervals of time in hours (e.g., 11:00 a.m. to 4:00 p.m.).

2.0 Students identify and describe the attributes of common figures in the plane and of common objects in space:

- 2.1** Describe and classify plane and solid geometric shapes (e.g., circle, triangle, square, rectangle, sphere, pyramid, cube, rectangular prism) according to the number and shape of faces, edges, and vertices.

Look at the pairs of shapes. Which is a pair of rectangles?
(CST released test question, 2004)



- 2.2** Put shapes together and take them apart to form other shapes (e.g., two congruent right triangles can be arranged to form a rectangle).

Statistics, Data Analysis, and Probability

1.0 Students collect numerical data and record, organize, display, and interpret the data on bar graphs and other representations:

- 1.1 Record numerical data in systematic ways, keeping track of what has been counted.
- 1.2 Represent the same data set in more than one way (e.g., bar graphs and charts with tallies).
- 1.3 Identify features of data sets (range and mode).
- 1.4 Ask and answer simple questions related to data representations.

2.0 Students demonstrate an understanding of patterns and how patterns grow and describe them in general ways:

- 2.1 Recognize, describe, and extend patterns and determine a next term in linear patterns (e.g., 4, 8, 12 . . . ; the number of ears on one horse, two horses, three horses, four horses).

If there are two horses on a farm, how many horseshoes will we need to shoe all the horses? Show, in an organized way, how many horseshoes we will need for 3, 4, 5, 6, 7, 8, 9, and 10 horses.

- 2.2 Solve problems involving simple number patterns.

Mathematical Reasoning

Grade Two

1.0 Students make decisions about how to set up a problem:

- 1.1 Determine the approach, materials, and strategies to be used.
- 1.2 Use tools, such as manipulatives or sketches, to model problems.

2.0 Students solve problems and justify their reasoning:

- 2.1 Defend the reasoning used and justify the procedures selected.
- 2.2 Make precise calculations and check the validity of the results in the context of the problem.

3.0 Students note connections between one problem and another.

Grade Three Mathematics Content Standards

By the end of grade three, students deepen their understanding of place value and their understanding of and skill with addition, subtraction, multiplication, and division of whole numbers. Students estimate, measure, and describe objects in space. They use patterns to help solve problems. They represent number relationships and conduct simple probability experiments.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified, particularly in the primary grades, before they are used with students.

The symbol ● identifies the key standards for grade three.

Number Sense

1.0 Students understand the place value of whole numbers:

- 1.1 Count, read, and write whole numbers to 10,000.

What is the smallest whole number you can make using the digits 4, 3, 9, and 1? Use each digit exactly once. (Adapted from TIMSS gr. 3–4, T-2)

- 1.2 Compare and order whole numbers to 10,000.

Which set of numbers is in order from greatest to least?
(CST released test question, 2004)

- (a) 147,163,234,275 (c) 275,163,234,147
(b) 275,234,163,147 (d) 163,275,234,147

- 1.3 Identify the place value for each digit in numbers to 10,000.

- 1.4 Round off numbers to 10,000 to the nearest ten, hundred, and thousand.

Round 9,582 to the nearest thousand.

- 1.5 Use expanded notation to represent numbers (e.g., $3,206 = 3,000 + 200 + 6$).

Sophie has 527 seashells in her collection. Which of these equals 527?
(CST released test question, 2004)

- (a) $5 + 2 + 7$ (c) $500 + 20 + 7$
(b) $5 + 20 + 700$ (d) $500 + 200 + 70$

2.0 Students calculate and solve problems involving addition, subtraction, multiplication, and division:

- 2.1 Find the sum or difference of two whole numbers between 0 and 10,000.

1. $562 + 27 = ?$
2. $5,286 + 2,845 = ?$
3. $3,215 - 2,876 = ?$

To prepare for recycling on Monday, Michael collected all the bottles in the house. He found 5 dark green ones, 8 clear ones with liquid still in them, 11 brown ones that used to hold root beer, 2 still with the cap on from his parents' cooking needs, and 4 more that were oversized. How many bottles did Michael collect? (This problem also supports Mathematical Reasoning Standard 1.1.)

- 2.2** Memorize to automaticity the multiplication table for numbers between 1 and 10.
- 2.3** Use the inverse relationship of multiplication and division to compute and check results.

Use multiplication to express 24 divided by 8 = 3.

John divided 135 by 5 and got 29 as his answer. Use multiplication to see if this division problem is solved correctly.

The figure shown below is a model for the multiplication sentence $8 \times 4 = 32$.



Which division sentence is modeled by the same figure?
(CST released test question, 2004)

- (a) $8 \div 4 = 2$ (b) $12 \div 4 = 3$ (c) $24 \div 8 = 3$ (d) $32 \div 8 = 4$

- 2.4** Solve simple problems involving multiplication of multidigit numbers by one-digit numbers ($3,671 \times 3 = \underline{\quad}$).
- 2.5 Solve division problems in which a multidigit number is evenly divided by a one-digit number ($135 \div 5 = \underline{\quad}$).
- 2.6 Understand the special properties of 0 and 1 in multiplication and division.

True or false?

- $24 \times 0 = 24$
- $19 \div 1 = 19$
- $63 \times 1 = 63$
- $0 \div 0 = 1$

- 2.7 Determine the unit cost when given the total cost and number of units.
- 2.8 Solve problems that require two or more of the skills mentioned above.

A price list in a store states: pen sets, \$3; magnets, \$4; sticker sets, \$6. How much would it cost to buy 5 pen sets, 7 magnets, and 8 sticker sets?

A tree was planted 54 years before 1961. How old is the tree in 1998?

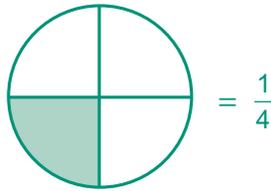
A class of 73 students go on a field trip. The school hires vans, each of which can seat a maximum of 10 students. The school policy is to seat as many students as possible in a van before using the next one. How many vans are needed?

3.0 Students understand the relationship between whole numbers, simple fractions, and decimals:

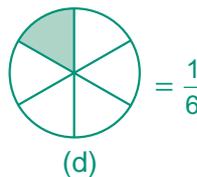
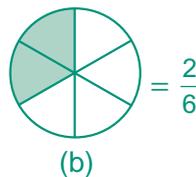
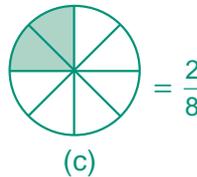
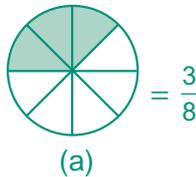
Grade Three

- 3.1 Compare fractions represented by drawings or concrete materials to show equivalency and to add and subtract simple fractions in context (e.g., $\frac{1}{2}$ of a pizza is the same amount as $\frac{2}{4}$ of another pizza that is the same size; show that $\frac{3}{8}$ is larger than $\frac{1}{4}$).

The circle shows $\frac{1}{4}$ shaded. (CST released test question, 2004)



Which fractional part of a circle below is equal to $\frac{1}{4}$?
(CST released test question, 2004)



- 3.2** Add and subtract simple fractions (e.g., determine that $\frac{1}{8} + \frac{3}{8}$ is the same as $\frac{1}{2}$).

Find the values:

- $\frac{1}{6} + \frac{2}{6} = ?$
- $\frac{7}{8} + \frac{3}{8} = ?$

- 3.3** Solve problems involving addition, subtraction, multiplication, and division of money amounts in decimal notation and multiply and divide money amounts in decimal notation by using whole-number multipliers and divisors.

Pedro bought 5 pens, 2 erasers and 2 boxes of crayons. The pens cost 65 cents each, the erasers 25 cents each, and a box of crayons \$1.10. The prices include tax, and Pedro paid with a ten-dollar bill. How much change did he get back?

- 3.4 Know and understand that fractions and decimals are two different representations of the same concept (e.g., 50 cents is $\frac{1}{2}$ of a dollar, 75 cents is $\frac{3}{4}$ of a dollar).

Grade Three

Algebra and Functions

1.0 Students select appropriate symbols, operations, and properties to represent, describe, simplify, and solve simple number relationships:

- 1.1** Represent relationships of quantities in the form of mathematical expressions, equations, or inequalities.

Write an inequality, equality, or expression to show each of the following relationships:

- 12 plus a number is less than 30.
- 4 times 6 is equal to 3 times a number.

Mr. Guzman bought 48 doughnuts packed equally into 4 boxes. Which number sentence shows how to find the number of doughnuts in each box? (CST released test question, 2004)

- (a) $48 - 4 = \underline{\quad}$ (c) $48 + 4 = \underline{\quad}$
 (b) $48 \div 4 = \underline{\quad}$ (d) $48 \times 4 = \underline{\quad}$

- 1.2 Solve problems involving numeric equations or inequalities.

If $6 + N > 9$, circle all the numbers that N could be 3, 2, 4, 1, 0, 8, 5.

What number makes this number sentence true $3 + 5 = \underline{\quad} \times 2$?
 (CST released test question, 2004)

- 1.3 Select appropriate operational and relational symbols to make an expression true (e.g., if $4 \underline{\quad} 3 = 12$, what operational symbol goes in the blank?).
- 1.4 Express simple unit conversions in symbolic form (e.g., $\underline{\quad}$ inches = $\underline{\quad}$ feet $\times 12$).

If number of feet = number of yards $\times 3$, and number of inches = number of feet $\times 12$, how many inches are there in 4 yards?

- 1.5 Recognize and use the commutative and associative properties of multiplication (e.g., if $5 \times 7 = 35$, then what is 7×5 ? and if $5 \times 7 \times 3 = 105$, then what is $7 \times 3 \times 5$?).

2.0 Students represent simple functional relationships:

- 2.1** Solve simple problems involving a functional relationship between two quantities (e.g., find the total cost of multiple items given the cost per unit).

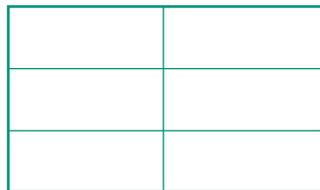
Grade Three

John wants to buy a dozen pencils. One store offers pencils at 6 for \$1. Another offers them at 4 for 65 cents. Yet another sells pencils at 15 cents each. Where should John purchase his pencils in order to save the most money?

One stamp costs 34¢. Two stamps cost 68¢. Three stamps cost \$1.02. If the cost of each stamp remains the same, how much would 4 stamps cost? (CST released test question, 2004)

- 2.2 Extend and recognize a linear pattern by its rules (e.g., the number of legs on a given number of horses may be calculated by counting by 4s or by multiplying the number of horses by 4).

Here is the beginning of a pattern of tiles. Assuming that each figure adds two more tiles to the preceding one, how many tiles will be in the sixth figure? (Adapted from TIMSS gr. 3–4, K-6)



Measurement and Geometry

1.0 Students choose and use appropriate units and measurement tools to quantify the properties of objects:

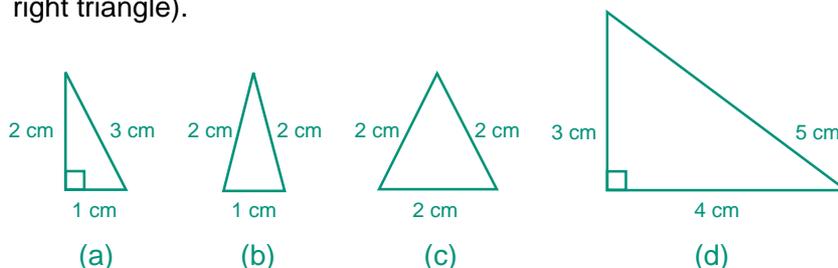
- 1.1 Choose the appropriate tools and units (metric and U.S.) and estimate and measure the length, liquid volume, and weight/mass of given objects.

- 1.2** Estimate or determine the area and volume of solid figures by covering them with squares or by counting the number of cubes that would fill them.
- 1.3** Find the perimeter of a polygon with integer sides.
- 1.4 Carry out simple unit conversions within a system of measurement (e.g., centimeters and meters, hours and minutes).

2.0 Students describe and compare the attributes of plane and solid geometric figures and use their understanding to show relationships and solve problems:

Grade Three

- 2.1** Identify, describe, and classify polygons (including pentagons, hexagons, and octagons).
- 2.2** Identify attributes of triangles (e.g., two equal sides for the isosceles triangle, three equal sides for the equilateral triangle, right angle for the right triangle).

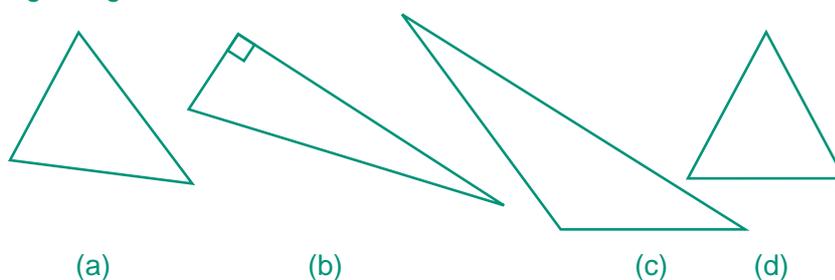


1. Which triangle has only 2 equal sides?
2. Which triangle has 3 equal sides?
3. Which triangle(s) have one right angle?

- 2.3** Identify attributes of quadrilaterals (e.g., parallel sides for the parallelogram, right angles for the rectangle, equal sides and right angles for the square).

- 2.4 Identify right angles in geometric figures or in appropriate objects and determine whether other angles are greater or less than a right angle.

Which of the following triangles include an angle that is greater than a right angle?



- 2.5 Identify, describe, and classify common three-dimensional geometric objects (e.g., cube, rectangular solid, sphere, prism, pyramid, cone, cylinder).
- 2.6 Identify common solid objects that are the components needed to make a more complex solid object.

Statistics, Data Analysis, and Probability

1.0 Students conduct simple probability experiments by determining the number of possible outcomes and make simple predictions:

- 1.1 Identify whether common events are certain, likely, unlikely, or improbable.

Are any of the following certain, likely, unlikely, or impossible?

1. Take two cubes each with the numbers 1, 2, 3, 4, 5, 6 written on its six faces. Throw them at random, and the sum of the numbers on the top faces is 12.
 2. It snows on New Year's Day.
 3. A baseball game is played somewhere in this country on any Sunday in July.
 4. It is sunny in June.
 5. Pick any two one-digit numbers, and their sum is 17.
- 1.2** Record the possible outcomes for a simple event (e.g., tossing a coin) and systematically keep track of the outcomes when the event is repeated many times.
 - 1.3** Summarize and display the results of probability experiments in a clear and organized way (e.g., use a bar graph or a line plot).
 - 1.4 Use the results of probability experiments to predict future events (e.g., use a line plot to predict the temperature forecast for the next day).

Mathematical Reasoning

1.0 Students make decisions about how to approach problems:

- 1.1 Analyze problems by identifying relationships, distinguishing relevant from irrelevant information, sequencing and prioritizing information, and observing patterns.
- 1.2 Determine when and how to break a problem into simpler parts.

2.0 Students use strategies, skills, and concepts in finding solutions:

- 2.1 Use estimation to verify the reasonableness of calculated results.

Prove or disprove a classmate's claim that 49 is more than 21 because 9 is more than 1.

- 2.2 Apply strategies and results from simpler problems to more complex problems.

- 2.3 Use a variety of methods, such as words, numbers, symbols, charts, graphs, tables, diagrams, and models, to explain mathematical reasoning.

- 2.4 Express the solution clearly and logically by using the appropriate mathematical notation and terms and clear language; support solutions with evidence in both verbal and symbolic work.

- 2.5 Indicate the relative advantages of exact and approximate solutions to problems and give answers to a specified degree of accuracy.

- 2.6 Make precise calculations and check the validity of the results from the context of the problem.

Grade Three**3.0 Students move beyond a particular problem by generalizing to other situations:**

- 3.1 Evaluate the reasonableness of the solution in the context of the original situation.

- 3.2 Note the method of deriving the solution and demonstrate a conceptual understanding of the derivation by solving similar problems.

- 3.3 Develop generalizations of the results obtained and apply them in other circumstances.

Grade Four Mathematics Content Standards

By the end of grade four, students understand large numbers and addition, subtraction, multiplication, and division of whole numbers. They describe and compare simple fractions and decimals. They understand the properties of, and the relationships between, plane geometric figures. They collect, represent, and analyze data to answer questions.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified, particularly in the primary grades, before they are used with students.

The symbol ● identifies the key standards for grade four.

Number Sense

1.0 Students understand the place value of whole numbers and decimals to two decimal places and how whole numbers and decimals relate to simple fractions. Students use the concepts of negative numbers:

- 1.1** Read and write whole numbers in the millions.
- Which of these is the number 5,005,014?
(CST released test question, 2004)
1. Five million, five hundred, fourteen
 2. Five million, five thousand, fourteen
 3. Five thousand, five hundred, fourteen
 4. Five billion, five million, fourteen
- 1.2** Order and compare whole numbers and decimals to two decimal places.
- Which is bigger: 3.1 or 3.09?
- 1.3** Round whole numbers through the millions to the nearest ten, hundred, thousand, ten thousand, or hundred thousand.
- Two hundred twenty-four students attend Green Street School. Round this number to the nearest hundred.
- Lunch was served to 3,778 students. Round this number to the nearest thousand.
- Each year it is estimated that 42,225 Canadian geese migrate south to warmer climates. Round this number to the nearest ten thousand.
- 1.4** Decide when a rounded solution is called for and explain why such a solution may be appropriate.
- Norberto has ten dollars and he wants to buy some ballpoint pens, which cost \$2.35; some notebooks, which cost \$4.40; and a fancy eraser, which costs \$1.45. He wants to make sure he has enough money to pay for all of them, so he rounds the cost of each item to the nearest dollar and adds them up: $\$2 + \$4 + \$1 = \7 . He concludes that his ten dollars would be sufficient to buy all the items. Is he correct and, if so, why? If the estimate that he makes turns out to be \$8 instead of \$7, should he be concerned?

- 1.5 Explain different interpretations of fractions, for example, parts of a whole, parts of a set, and division of whole numbers by whole numbers; explain equivalence of fractions (see Standard 4.0).
- 1.6 Write tenths and hundredths in decimal and fraction notations and know the fraction and decimal equivalents for halves and fourths (e.g., $\frac{1}{2} = 0.5$ or 0.50 ; $\frac{7}{4} = 1\frac{3}{4} = 1.75$).

Which fraction means the same as 0.17?

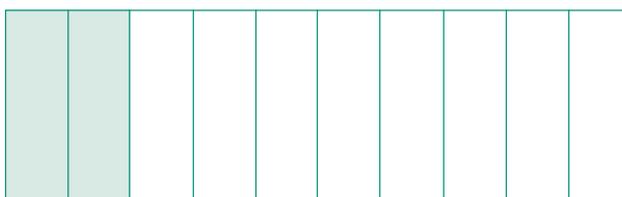
- (a) $\frac{17}{10}$ (b) $\frac{17}{100}$ (c) $\frac{17}{1000}$ (d) $\frac{17}{1}$

(CST released test question, 2004)

- 1.7 Write the fraction represented by a drawing of parts of a figure; represent a given fraction by using drawings; and relate a fraction to a simple decimal on a number line.

Which number represents the shaded part of the figure?

(Adapted from TIMSS gr. 3–4, M-5)



- (a) 2.8 (c) 0.2
(b) 0.5 (d) 0.02

- 1.8** Use concepts of negative numbers (e.g., on a number line, in counting, in temperature, in “owing”).

Yesterday’s temperature was 5 degrees Celsius, but the temperature dropped 9 degrees Celsius overnight. What is today’s temperature?

Determine if the following number sentences are true or false by identifying the relative positions of each number on a number line:

$$-9 > -10 \qquad -31 < -29$$

- 1.9** Identify on a number line the relative position of positive fractions, positive mixed numbers, and positive decimals to two decimal places.

Write a positive number for each letter on the number line shown below.



Which letter represents the number closest to 2.5?

Write the letter that represents where each number would go on the number line shown below:

$$1\frac{1}{4} \quad \underline{\hspace{1cm}} \qquad 2.50 \quad \underline{\hspace{1cm}} \qquad \frac{3}{4} \quad \underline{\hspace{1cm}}$$



Grade Four

Determine if the following number sentences are true or false by identifying the relative positions of each number on a number line:

1. $\frac{1}{4} > 2.54$
2. $\frac{5}{2} < 2.6$
3. $\frac{12}{18} = \frac{2}{3}$ (Note the equivalence of fractions.)
4. $\frac{4}{5} < \frac{13}{15}$

2.0 Students extend their use and understanding of whole numbers to the addition and subtraction of simple decimals:

- 2.1 Estimate and compute the sum or difference of whole numbers and positive decimals to two places.

$$\text{Solve } 55.73 - 48.25 = ?$$

- 2.2 Round two-place decimals to one decimal or to the nearest whole number and judge the reasonableness of the rounded answer.

In her science class Li Ping weighs two samples of quartz and determines that the first has a weight of 3.44 grams and the second has a weight of 2.39 grams. Her teacher wants Li Ping to report the combined weight of the two samples to the nearest tenth of a gram, and to the nearest gram; however, the scale cannot measure weights over 5 grams. Li Ping decides to round the numbers first and then to add them.

1. Is $3.4 + 2.4$ a reasonable estimate of the combined weights to the nearest tenth of a gram?
2. Is $3 + 2$ a reasonable estimate of the combined weights to the nearest gram?

3.0 Students solve problems involving addition, subtraction, multiplication, and division of whole numbers and understand the relationships among the operations:

- 3.1** Demonstrate an understanding of, and the ability to use, standard algorithms for the addition and subtraction of multidigit numbers.

Solve these problems using the standard algorithms:

- $619,581 - 23,183 = ?$
- $6,747 + 321,105 = ?$

- 3.2** Demonstrate an understanding of, and the ability to use, standard algorithms for multiplying a multidigit number by a two-digit number and for dividing a multidigit number by a one-digit number; use relationships between them to simplify computations and to check results.

Singh and Sepideh work independently to solve the problem $783 \times 23 = ?$ They apply slightly different approaches, as shown below. Explain why both approaches are valid and give the same answer.

Grade Four

$\begin{array}{r} 783 \\ \times 3 \\ \hline 2,349 \end{array}$	$\begin{array}{r} 783 \\ \times 20 \\ \hline 15,660 \end{array}$	$\begin{array}{r} 2,349 \\ + 15,660 \\ \hline 18,009 \end{array}$	$\begin{array}{r} 783 \\ \times 23 \\ \hline 2,349 \\ + 15,660 \\ \hline 18,009 \end{array}$
Singh			Sepideh

- 3.3** Solve problems involving multiplication of multidigit numbers by two-digit numbers.
- 3.4** Solve problems involving division of multidigit numbers by one-digit numbers.

Solve each of the following problems and observe the different roles played by the number 37 in each situation:

- Four children shared 37 dollars equally. How much did each get?
- Four children shared 37 pennies as equally as possible. How many pennies did each get?
- Cars need to be rented for 37 children going on a field trip. Each car can take 12 children in addition to the driver. How many cars must be rented?
- There are 9 rows of seats in a theater. Each row has the same number of seats. If there is a total of 162 seats, how many seats are in each row? (CST released test question, 2004)

4.0 Students know how to factor small whole numbers:

- 4.1 Understand that many whole numbers break down in different ways (e.g., $12 = 4 \times 3 = 2 \times 6 = 2 \times 2 \times 3$).

In how many distinct ways can you write 60 as a product of two numbers?

- 4.2** Know that numbers such as 2, 3, 5, 7, and 11 do not have any factors except 1 and themselves and that such numbers are called prime numbers.

Circle all the prime numbers in these different representations of 24:

- (a) 2×12 (c) 4×6 (e) $2 \times 3 \times 4$ (g) 1×24
 (b) 3×8 (d) $2 \times 2 \times 6$ (f) $2 \times 2 \times 2 \times 3$

Grade Four

Algebra and Functions

1.0 Students use and interpret variables, mathematical symbols, and properties to write and simplify expressions and sentences:

- 1.1 Use letters, boxes, or other symbols to stand for any number in simple expressions or equations (e.g., demonstrate an understanding and the use of the concept of a variable).

Tanya has read the first 78 pages of a 130-page book. Give the number sentence that can be used to find the number of pages Tanya must read to finish the book. (Adapted from TIMSS gr. 3–4, I-7)

1. $130 + 78 = \underline{\quad}$
2. $\underline{\quad} - 78 \square = 130$
3. $130 - 78 = \underline{\quad}$
4. $130 - \underline{\quad} = 178$

- 1.2** Interpret and evaluate mathematical expressions that now use parentheses.

Evaluate the two expressions:

$$(28 - 10) - 8 = \underline{\quad} \text{ and } 28 - (10 - 8) = \underline{\quad}.$$

Solve $5 \times (8 - 2) = ?$ (CST released test question, 2004)

- 1.3** Use parentheses to indicate which operation to perform first when writing expressions containing more than two terms and different operations.

What is the value of the expression below?

$$(13 + 4) - (7 \times 2) + (31 - 17)$$

(Adapted from CST released test question, 2004)

- 1.4 Use and interpret formulas (e.g., area = length \times width or $A = lw$) to answer questions about quantities and their relationships.

Vik has a car that has a 16-gallon gas tank. When the tank is filled, he can drive 320 miles before running out of gas. How can Vik calculate his car's mileage in miles/gallon?

- 1.5** Understand that an equation such as $y = 3x + 5$ is a prescription for determining a second number when a first number is given.

2.0 Students know how to manipulate equations:

- 2.1** Know and understand that equals added to equals are equal.

The letters S and T stand for numbers. If $S - 100 = T - 100$, which statement is true? (CST released test question, 2004)

$$S = T \qquad S > T \qquad S = T + 100 \qquad S > T + 100$$

Grade Four

- 2.2** Know and understand that equals multiplied by equals are equal.

What number goes into the box to make this number sentence true?
 $(7 - 3) \times 5 = 4 \times [\text{box}]$ (CST released test question, 2004)

Measurement and Geometry

1.0 Students understand perimeter and area:

- 1.1 Measure the area of rectangular shapes by using appropriate units, such as square centimeter (cm^2), square meter (m^2), square kilometer (km^2), square inch (in.^2), square yard (yd.^2), or square mile (mi.^2).
- 1.2 Recognize that rectangles that have the same area can have different perimeters.

Draw a rectangle whose area is 120 and whose perimeter exceeds 50. Draw another rectangle with the same area whose perimeter exceeds 240.

- 1.3 Understand that rectangles that have the same perimeter can have different areas.

Is the area of a 45×55 rectangle (in cm^2) smaller or bigger than that of a square with the same perimeter?

Draw a rectangle whose perimeter is 40 and whose area is less than 20.

- 1.4 Understand and use formulas to solve problems involving perimeters and areas of rectangles and squares. Use those formulas to find the areas of more complex figures by dividing the figures into basic shapes.

The length of a rectangle is 6 cm, and its perimeter is 16 cm. What is the area of the rectangle in square centimeters? (TIMSS gr. 7–8, K-5)

2.0 Students use two-dimensional coordinate grids to represent points and graph lines and simple figures:

- 2.1** Draw the points corresponding to linear relationships on graph paper (e.g., draw 10 points on the graph of the equation $y = 3x$ and connect them by using a straight line).

1. Draw ten points on the graph of the equation $x = 4$.
2. Draw ten points on the graph of the equation $y = 71$.
3. Draw ten points on the graph of the equation $y = 2x + 4$.

- 2.2** Understand that the length of a horizontal line segment equals the difference of the x -coordinates.

What is the length of the line segment joining the points $(6, -4)$ and $(21, -4)$?

- 2.3** Understand that the length of a vertical line segment equals the difference of the y -coordinates.

What is the length of the line segment joining the points $(121, 3)$ to $(121, 17)$?

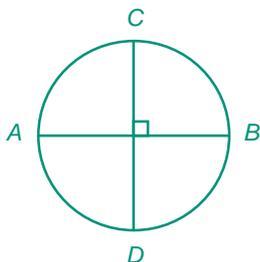
3.0 Students demonstrate an understanding of plane and solid geometric objects and use this knowledge to show relationships and solve problems:

- 3.1 Identify lines that are parallel and perpendicular.
(Teachers are advised to introduce the terms *intersecting lines* and *nonintersecting lines* when dealing with this standard.)
- 3.2 Identify the radius and diameter of a circle.
- 3.3 Identify congruent figures.
- 3.4 Identify figures that have bilateral and rotational symmetry.

Craig folded a piece of paper in half and cut out a shape along the folded edge. Draw a picture to show what the cutout shape will look like when it is opened up and flattened out.

(Adapted from TIMSS gr. 3–4, T-5)

Let AB , CD be perpendicular diameters of a circle, as shown. If we reflect across the line segment CD , what happens to A and what happens to B under this reflection?



Grade Four

- 3.5 Know the definitions of a right angle, an acute angle, and an obtuse angle. Understand that 90° , 180° , 270° , and 360° are associated, respectively, with $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and full turns.
- 3.6 Visualize, describe, and make models of geometric solids (e.g., prisms, pyramids) in terms of the number and shape of faces, edges, and vertices; interpret two-dimensional representations of three-dimensional objects; and draw patterns (of faces) for a solid that, when cut and folded, will make a model of the solid.
- 3.7 Know the definitions of different triangles (e.g., equilateral, isosceles, scalene) and identify their attributes.

Name each of the following triangles:

1. No equal sides
 2. Two equal sides
 3. Three equal sides
- 3.8 Know the definition of different quadrilaterals (e.g., rhombus, square, rectangle, parallelogram, trapezoid).

Explain which of the following statements are true and why:

1. All squares are rectangles.
2. All rectangles are squares.
3. All parallelograms are rectangles.
4. All rhombi are parallelograms.
5. Some parallelograms are squares.

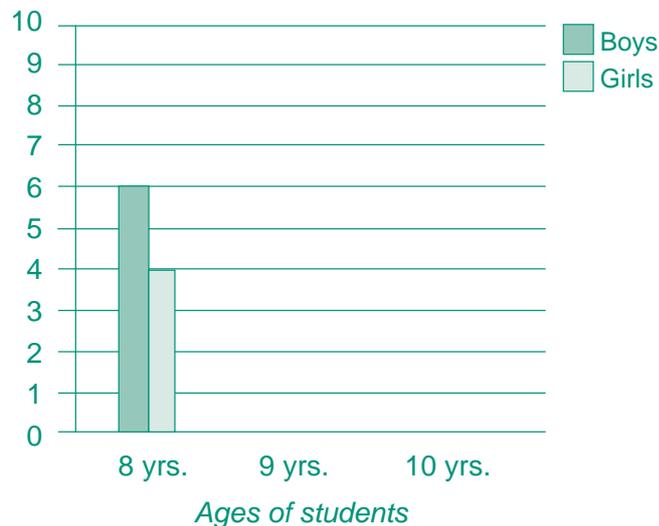
Statistics, Data Analysis, and Probability

1.0 Students organize, represent, and interpret numerical and categorical data and clearly communicate their findings:

The following table shows the ages of the girls and boys in a club. Complete the graph by using the information for ages 9 and 10 shown in the table. (Adapted from TIMSS gr. 3–4, S-1)

Grade Four

Ages	Number of Girls	Number of Boys
8	4	6
9	8	4
10	6	10



- 1.1 Formulate survey questions; systematically collect and represent data on a number line; and coordinate graphs, tables, and charts.
- 1.2 Identify the mode(s) for sets of categorical data and the mode(s), median, and any apparent outliers for numerical data sets.
- 1.3 Interpret one- and two-variable data graphs to answer questions about a situation.

2.0 Students make predictions for simple probability situations:

Nine identical chips numbered 1 through 9 are put in a jar. When a chip is drawn from the jar, what is the probability that it has an even number? (Adapted from TIMSS gr. 7–8, N-18)

- 2.1 Represent all possible outcomes for a simple probability situation in an organized way (e.g., tables, grids, tree diagrams).

- 2.2 Express outcomes of experimental probability situations verbally and numerically (e.g., 3 out of 4; $\frac{3}{4}$).

Royce has a bag with 8 red marbles, 4 blue marbles, 5 green marbles, and 9 yellow marbles all the same size. If he pulls out 1 marble without looking, which color is he most likely to choose? (CST released test question, 2004)

Mathematical Reasoning

1.0 Students make decisions about how to approach problems:

- 1.1 Analyze problems by identifying relationships, distinguishing relevant from irrelevant information, sequencing and prioritizing information, and observing patterns.
- 1.2 Determine when and how to break a problem into simpler parts.

2.0 Students use strategies, skills, and concepts in finding solutions:

- 2.1 Use estimation to verify the reasonableness of calculated results.
- 2.2 Apply strategies and results from simpler problems to more complex problems.
- 2.3 Use a variety of methods, such as words, numbers, symbols, charts, graphs, tables, diagrams, and models, to explain mathematical reasoning.
- 2.4 Express the solution clearly and logically by using the appropriate mathematical notation and terms and clear language; support solutions with evidence in both verbal and symbolic work.
- 2.5 Indicate the relative advantages of exact and approximate solutions to problems and give answers to a specified degree of accuracy.
- 2.6 Make precise calculations and check the validity of the results from the context of the problem.

3.0 Students move beyond a particular problem by generalizing to other situations:

- 3.1 Evaluate the reasonableness of the solution in the context of the original situation.
- 3.2 Note the method of deriving the solution and demonstrate a conceptual understanding of the derivation by solving similar problems.
- 3.3 Develop generalizations of the results obtained and apply them in other circumstances.

Grade Five Mathematics Content Standards

By the end of grade five, students increase their facility with the four basic arithmetic operations applied to fractions and decimals and learn to add and subtract positive and negative numbers. They know and use common measuring units to determine length and area and know and use formulas to determine the volume of simple geometric figures. Students know the concept of angle measurement and use a protractor and compass to solve problems. They use grids, tables, graphs, and charts to record and analyze data.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified, particularly in the primary grades, before they are used with students.

The symbol ● identifies the key standards for grade five.

Number Sense

1.0 Students compute with very large and very small numbers, positive integers, decimals, and fractions and understand the relationship between decimals, fractions, and percents. They understand the relative magnitudes of numbers:

1.1 Estimate, round, and manipulate very large (e.g., millions) and very small (e.g., thousandths) numbers.

● 1.2 Interpret percents as a part of a hundred; find decimal and percent equivalents for common fractions and explain why they represent the same value; compute a given percent of a whole number.

What is 40% of 250? (CST released test question, 2004)

A test had 48 problems. Joe got 42 correct.

1. What percent were correct?
2. What percent were wrong?
3. If Moe got 93.75% correct, how many problems did he get correct?

1.3 Understand and compute positive integer powers of nonnegative integers; compute examples as repeated multiplication.

Which is bigger: 3^5 or 5^3 ?

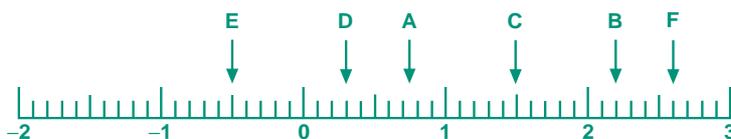
● 1.4 Determine the prime factors of all numbers through 50 and write the numbers as the product of their prime factors by using exponents to show multiples of a factor (e.g., $24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$).

Find the prime factorization of 48 and use exponents where appropriate.

● 1.5 Identify and represent on a number line decimals, fractions, mixed numbers, and positive and negative integers.

Next to each number, write the letter that represents the quantity on the number line.

2.2 _____ 0.3 _____ -0.5 _____
 $2\frac{6}{10}$ _____ $\frac{75}{100}$ _____ 1.5 _____



Place the following numbers, in approximate positions, on the number line: $1\frac{3}{7}$, 1.43, $\frac{23}{14}$.

2.0 Students perform calculations and solve problems involving addition, subtraction, and simple multiplication and division of fractions and decimals:

Grade Five

- 2.1** Add, subtract, multiply, and divide with decimals; add with negative integers; subtract positive integers from negative integers; and verify the reasonableness of the results.

Determine the following numbers:

1. $11 + (-23)$
2. $(-15) - 128$
3. $51 - 24.7$
4. 8.2×24.7
5. $68.13 \div 3$

- 2.2** Demonstrate proficiency with division, including division with positive decimals and long division with multidigit divisors.

Find the quotient: 6 divided by 0.025.

$15.12 \div 2.4 = ?$ (CST released test question, 2004)

- 2.3** Solve simple problems, including ones arising in concrete situations, involving the addition and subtraction of fractions and mixed numbers (like and unlike denominators of 20 or less), and express answers in the simplest form.

Suppose a galleon is a type of money worth 17 sickles. If Ludo borrows $2\frac{3}{17}$ galleons from Harry, then gives him back 12 sickles, how many galleons and sickles does Ludo still owe?

Sally is training to walk in a marathon. In her second week of training, she walked $5\frac{3}{4}$ miles on Tuesday, $5\frac{1}{16}$ miles on Thursday, and $16\frac{3}{8}$ miles on Sunday. How many miles altogether did Sally walk on those three days?

Jerry and Larry both ordered personal-sized pizzas for lunch. Jerry ate $\frac{3}{4}$ of his pizza, and Larry ate $\frac{2}{3}$ of his pizza. Who ate more pizza and how much more did he eat?

Given the following three pairs of fractions: $\frac{3}{8}$ and $\frac{1}{16}$, $5\frac{1}{4}$ and $1\frac{3}{4}$, 16 and $3\frac{1}{5}$, find for each pair its:

1. Sum
2. Difference

- 2.4 Understand the concept of multiplication and division of fractions.

$$\frac{3}{4} \div \frac{3}{5} = ? \text{ (CST released test question, 2004)}$$

- 2.5 Compute and perform simple multiplication and division of fractions and apply these procedures to solving problems.

Given the following three pairs of fractions: $\frac{3}{8}$ and $\frac{1}{16}$, $5\frac{1}{4}$ and $1\frac{3}{4}$, 16 and $3\frac{1}{5}$, find for each pair its:

1. Product
2. Quotient in simplest terms

Ericka has $3\frac{1}{2}$ yards of cloth to make shirts. Each shirt requires $\frac{7}{8}$ yard. How many shirts can she make? How much cloth will she have left over?

Algebra and Functions

- 1.0 Students use variables in simple expressions, compute the value of the expression for specific values of the variable, and plot and interpret the results:**

- 1.1 Use information taken from a graph or equation to answer questions about a problem situation.

- 1.2** Use a letter to represent an unknown number; write and evaluate simple algebraic expressions in one variable by substitution.

If x is a number that satisfies $3x + 2 = 14$, can x be equal to 3?

If $N = 4$, what is the value of $6 \times N - 3$?

(CST released test question, 2004)

- 1.3 Know and use the distributive property in equations and expressions with variables.

What value for z makes this equation true?

$$8 \times 37 = (8 \times 30) + (8 \times z) \text{ (CST released test question, 2004)}$$

- 1.4** Identify and graph ordered pairs in the four quadrants of the coordinate plane.

Plot these points on a coordinate plane:

$(1, 2)$, $(-4, -3)$, $(12, -1)$, $(0, 4)$, $(-4, 0)$

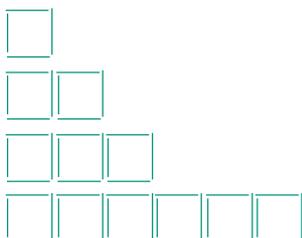
- 1.5** Solve problems involving linear functions with integer values; write the equation; and graph the resulting ordered pairs of integers on a grid.

Which equation could have been used to create this function table?
(CST released test question, 2004)

x	y
-9	-5
-2	2
4	8
11	15

$$y = \frac{x}{2} \quad y = 2x \quad y = x - 4 \quad y = x + 4$$

One can build rows of squares with toothpicks, as shown below for the case of 1, 2, 3, and 6 squares, respectively:



Explain why the following formula

$$y = 3n + 1$$

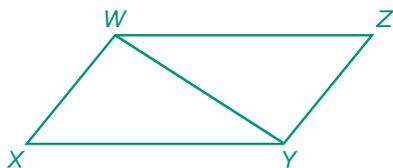
for the number of toothpicks y needed to form a row of n squares is correct. Graph this equation on a grid and remember that n takes on only whole number values 1, 2, 3, 4, . . .

Measurement and Geometry

- 1.0** Students understand and compute the volumes and areas of simple objects:

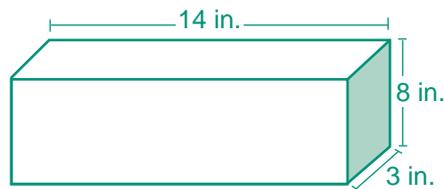
- 1.1** Derive and use the formula for the area of a triangle and of a parallelogram by comparing each with the formula for the area of a rectangle (i.e., two of the same triangles make a parallelogram with twice the area; a parallelogram is compared with a rectangle of the same area by pasting and cutting a right triangle on the parallelogram).

In the figure below, $WXYZ$ is a parallelogram.



If the area of triangle WXY is 22 square inches, what is the area of $WXYZ$? (CST released test question, 2004)

- 1.2** Construct a cube and rectangular box from two-dimensional patterns and use these patterns to compute the surface area for these objects.
- 1.3** Understand the concept of volume and use the appropriate units in common measuring systems (i.e., cubic centimeter [cm^3], cubic meter [m^3], cubic inch [in.^3], cubic yard [yd.^3]) to compute the volume of rectangular solids.

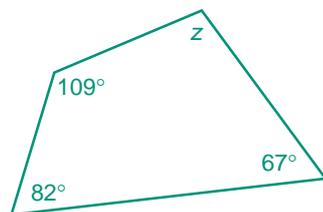


This rectangular prism has a length of 14 inches, a height of 8 inches, and a width of 3 inches. What is the volume? (CST released test question, 2004)

- 1.4 Differentiate between, and use appropriate units of measures for, two- and three-dimensional objects (i.e., find the perimeter, area, volume).

2.0 Students identify, describe, and classify the properties of, and the relationships between, plane and solid geometric figures:

- 2.1** Measure, identify, and draw angles, perpendicular and parallel lines, rectangles, and triangles by using appropriate tools (e.g., straightedge, ruler, compass, protractor, drawing software).
- 2.2** Know that the sum of the angles of any triangle is 180° and the sum of the angles of any quadrilateral is 360° and use this information to solve problems.



What is the measure of angle z in the figure above? (CST released test question, 2004)

- 2.3 Visualize and draw two-dimensional views of three-dimensional objects made from rectangular solids.

Statistics, Data Analysis, and Probability

1.0 Students display, analyze, compare, and interpret different data sets, including data sets of different sizes:

- 1.1 Know the concepts of mean, median, and mode; compute and compare simple examples to show that they may differ.

Compute the mean, median, and mode of the following collection of 27 numbers:

$1, 1, \dots, 1, 2, 3, 26, 135$
23

Grade Five

- 1.2 Organize and display single-variable data in appropriate graphs and representations (e.g., histogram, circle graphs) and explain which types of graphs are appropriate for various data sets.
- 1.3 Use fractions and percentages to compare data sets of different sizes.
- 1.4 Identify ordered pairs of data from a graph and interpret the meaning of the data in terms of the situation depicted by the graph.
- 1.5 Know how to write ordered pairs correctly; for example, (x, y) .

Mathematical Reasoning

1.0 Students make decisions about how to approach problems:

- 1.1 Analyze problems by identifying relationships, distinguishing relevant from irrelevant information, sequencing and prioritizing information, and observing patterns.
- 1.2 Determine when and how to break a problem into simpler parts.

2.0 Students use strategies, skills, and concepts in finding solutions:

- 2.1 Use estimation to verify the reasonableness of calculated results.
- 2.2 Apply strategies and results from simpler problems to more complex problems.
- 2.3 Use a variety of methods, such as words, numbers, symbols, charts, graphs, tables, diagrams, and models, to explain mathematical reasoning.
- 2.4 Express the solution clearly and logically by using the appropriate mathematical notation and terms and clear language; support solutions with evidence in both verbal and symbolic work.

- 2.5 Indicate the relative advantages of exact and approximate solutions to problems and give answers to a specified degree of accuracy.
- 2.6 Make precise calculations and check the validity of the results from the context of the problem.

3.0 Students move beyond a particular problem by generalizing to other situations:

- 3.1 Evaluate the reasonableness of the solution in the context of the original situation.
- 3.2 Note the method of deriving the solution and demonstrate a conceptual understanding of the derivation by solving similar problems.
- 3.3 Develop generalizations of the results obtained and apply them in other circumstances.

Grade Six Mathematics Content Standards

By the end of grade six, students have mastered the four arithmetic operations with whole numbers, positive fractions, positive decimals, and positive and negative integers; they accurately compute and solve problems. They apply their knowledge to statistics and probability. Students understand the concepts of mean, median, and mode of data sets and how to calculate the range. They analyze data and sampling processes for possible bias and misleading conclusions; they use addition and multiplication of fractions routinely to calculate the probabilities for compound events. Students conceptually understand and work with ratios and proportions; they compute percentages (e.g., tax, tips, interest). Students know about π and the formulas for the circumference and area of a circle. They use letters for numbers in formulas involving geometric shapes and in ratios to represent an unknown part of an expression. They solve one-step linear equations.

Number Sense

1.0 Students compare and order positive and negative fractions, decimals, and mixed numbers. Students solve problems involving fractions, ratios, proportions, and percentages:

- 1.1 Compare and order positive and negative fractions, decimals, and mixed numbers and place them on a number line.

Order the following numbers: $\frac{20}{21}$ $-\frac{4}{9}$ -4.4 $1\frac{1}{12}$ 1.1 $\frac{3}{7}$

If you were to place $-\frac{2}{3}$, -3 , and $-\frac{7}{8}$ on a number line, which number would be closest to -1 ? Use a number line to explain your answer.

Place the following numbers on a number line:

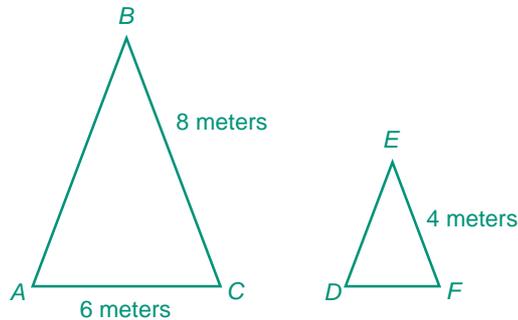
0.3 $-\frac{3}{10}$ $2\frac{1}{2}$ $\frac{4}{5}$ $\frac{7}{8}$ -2

- 1.2 Interpret and use ratios in different contexts (e.g., batting averages, miles per hour) to show the relative sizes of two quantities, using appropriate notations (a/b , a to b , $a:b$).
- 1.3 Use proportions to solve problems (e.g., determine the value of N if $\frac{4}{7} = \frac{N}{21}$, find the length of a side of a polygon similar to a known polygon). Use cross-multiplication as a method for solving such problems, understanding it as the multiplication of both sides of an equation by a multiplicative inverse.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified, particularly in the primary grades, before they are used with students.

The symbol ● identifies the key standards for grade six.

$\triangle ABC$ is similar to $\triangle DEF$. What is the length of \overline{DF} ?
(CST released test question, 2004)



Ballpoint pens are sold in bundles of four. Lee bought 24 pens for \$14.40. How much would 56 pens cost? Carefully explain your solution.

Find n if:

$$1. \frac{49}{21} = \frac{14}{n}$$

$$2. \frac{28}{n} = \frac{36}{27}$$

(This problem also applies to Algebra and Functions Standard 1.1.)

- 1.4** Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

Ann paid \$70.20 for a dress, and the amount includes an 8% sales tax. What is the cost of the dress before the tax?

- 2.0** Students calculate and solve problems involving addition, subtraction, multiplication, and division:

- 2.1 Solve problems involving addition, subtraction, multiplication, and division of positive fractions and explain why a particular operation was used for a given situation.

Your after-school program is on a hiking trip. You hike $\frac{3}{4}$ of a mile and stop to rest. Your friend hikes $\frac{4}{5}$ of a mile, then turns around and hikes back $\frac{1}{8}$ of a mile. Who is farther ahead on the trail? How much farther? Explain how you solved the problem.

At soccer practice the team has to run around a rectangular field that is $75\frac{1}{2}$ feet by $127\frac{3}{4}$ feet. The coach makes the team run around the field three times. How many total feet does a team member run? Explain how you solved this problem.

Mario wants to make half of his special no-bake cookie recipe. The recipe calls for $1\frac{3}{4}$ cups of white sugar, $\frac{1}{3}$ cup of margarine, $\frac{1}{2}$ cup of

peanut butter, and $3\frac{1}{4}$ cups of oats. How much of each ingredient will Mario need? Explain how you solved this problem.

Jim was on a hiking trail and after walking $\frac{3}{4}$ of a mile, he found that he was only $\frac{5}{8}$ of the way to the end of the trail. How long is the trail? Explain.

- 2.2 Explain the meaning of multiplication and division of positive fractions and perform the calculations (e.g., $\frac{5}{8} \div \frac{15}{16} = \frac{5}{8} \times \frac{16}{15} = \frac{2}{3}$).

Draw a picture that illustrates each of the following problems and its solution. Explain how your drawings illustrate the problems and the solutions.

1. $\frac{3}{4} \times \frac{1}{2}$

2. $\frac{3}{4} \div \frac{1}{2}$

3. $2 \times \frac{1}{4}$

- 2.3 Solve addition, subtraction, multiplication, and division problems, including those arising in concrete situations, that use positive and negative integers and combinations of these operations.

Two friends start out on a daylong hike. They start at an elevation of 526 feet. The morning hike takes them to an altitude 300 feet higher than where they started. In the afternoon the friends descend 117 feet and stop to rest. Then they continue downward and descend another 366 feet. Describe the change in altitude.

Simplify to make the calculation as simple as possible:

1. $-19 + 37 + 19$

2. $(-16)(-28) + (-16)27$

3. $(-8)(-4)(19)(6 + (-6))$

- 2.4 Determine the least common multiple and the greatest common divisor of whole numbers; use them to solve problems with fractions (e.g., to find a common denominator to add two fractions or to find the reduced form for a fraction).

$\frac{3}{8} + \frac{1}{2} = ?$ (CST released test question, 2004)

Algebra and Functions

1.0 Students write verbal expressions and sentences as algebraic expressions and equations; they evaluate algebraic expressions, solve simple linear equations, and graph and interpret their results:

- 1.1** Write and solve one-step linear equations in one variable.

What value of k makes the following equation true?

$$k \div 3 = 36 \text{ (CST released test question, 2004)}$$

$$y - 2 = 10. \text{ What is } y?$$

$$6y = 12. \text{ What is } y?$$

If a number y satisfies $y + 17 = 10$, what is y ? If a number x satisfies $3x = 25$, what is x ?

- 1.2 Write and evaluate an algebraic expression for a given situation, using up to three variables.

A telephone company charges \$0.05 per minute for local calls and \$0.12 per minute for long-distance calls. Which expression gives the total cost in dollars for x minutes of local calls and y minutes of long-distance calls? (CST released test question, 2004)

- (a) $0.05x + 0.12y$ (c) $0.17(x + y)$
(b) $0.05x - 0.12y$ (d) $0.17xy$

- 1.3 Apply algebraic order of operations and the commutative, associative, and distributive properties to evaluate expressions; and justify each step in the process.

Simplify:

$$1. (4^3 + 7) - (5 - 8)^3$$

$$2. 11[5(7^2) - 3^2 - 12(20 + 5.4 + 2)]$$

$$3. -3 \cdot (3^2 + 3) \div 3^2$$

- 1.4 Solve problems manually by using the correct order of operations or by using a scientific calculator.

2.0 Students analyze and use tables, graphs, and rules to solve problems involving rates and proportions:

- 2.1 Convert one unit of measurement to another (e.g., from feet to miles, from centimeters to inches).

Suppose that one British pound is worth \$1.50. In London a magazine costs 3 pounds. In San Francisco the same magazine costs \$4.25. In which city is the magazine cheaper?

When temperature is measured in both Celsius (C) and Fahrenheit (F), it is known that they are related by the following formula:

$9 \times C = (F - 32) \times 5$. What is 50 degrees Fahrenheit in Celsius?

(Note the explicit use of parentheses.)

How many inches are in $2\frac{1}{2}$ feet? (CST released test question, 2004)

- 2.2** Demonstrate an understanding that *rate* is a measure of one quantity per unit value of another quantity.

Joe can type 9 words in 8 seconds. At this rate, how many words can he type in 2 minutes?

- 2.3 Solve problems involving rates, average speed, distance, and time.

Marcus took a train from San Francisco to San Jose, a distance of 54 miles. The train took 45 minutes for the trip. What was the average speed of the train?

3.0 Students investigate geometric patterns and describe them algebraically:

- 3.1 Use variables in expressions describing geometric quantities (e.g., $P = 2w + 2l$, $A = \frac{1}{2}bh$, $C = \pi d$ —the formulas for the perimeter of a rectangle, the area of a triangle, and the circumference of a circle, respectively).

A rectangle has width w . Its length is one more than 3 times its width. Find the perimeter of the rectangle. (Your answer will be expressed in terms of w .)

- 3.2 Express in symbolic form simple relationships arising from geometry.

The rectangle shown below has length 15 inches and perimeter P inches.



Which equation could be used to find the width of the rectangle?

$$P = 15 + \frac{W}{2} \quad P = 15 - w \quad P = 30 + 2w \quad P = 30 - 2w$$

(CST released test question, 2004)

Measurement and Geometry

1.0 Students deepen their understanding of the measurement of plane and solid shapes and use this understanding to solve problems:

- 1.1** Understand the concept of a constant such as π ; know the formulas for the circumference and area of a circle.

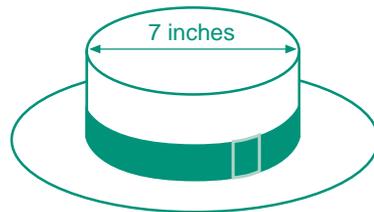
Which equation could be used to find the area in square inches of a circle with a radius of 8 inches? (CST released test question, 2004)

- (a) $A = 4 \times \pi$ (b) $A = \pi \times 4^2$ (c) $A = 8 \times \pi$ (d) $A = \pi \times 8^2$

- 1.2 Know common estimates of π (3.14; $\frac{22}{7}$) and use these values to estimate and calculate the circumference and the area of circles; compare with actual measurements.

What is the circumference of a circle with a radius of 5?
(Answer: 10π or approximately 31.4)

The top part of this hat is shaped like a cylinder with a diameter of 7 inches.

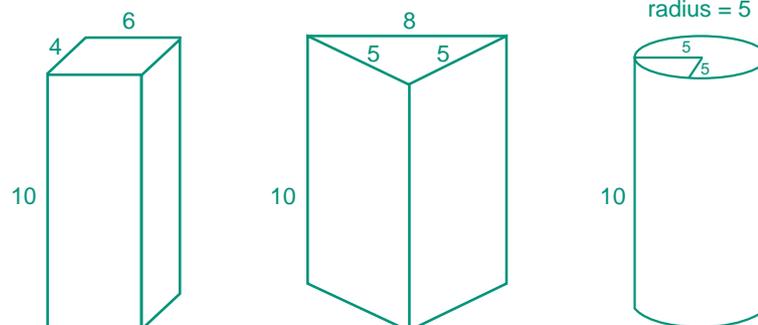


Which measure is *closest* to the length of the band that goes around the outside of the hat? (CST released test question, 2004)

- (a) 10.1 inches (b) 11.0 inches (c) 22.0 inches (d) 38.5 inches

- 1.3 Know and use the formulas for the volume of triangular prisms and cylinders (area of base \times height); compare these formulas and explain the similarity between them and the formula for the volume of a rectangular solid.

Find the volumes (dimensions are in cm).

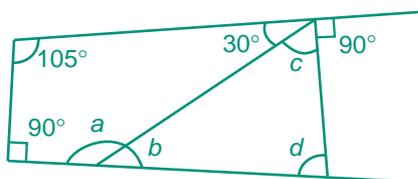


2.0 Students identify and describe the properties of two-dimensional figures:

2.1 Identify angles as vertical, adjacent, complementary, or supplementary and provide descriptions of these terms.

2.2 Use the properties of complementary and supplementary angles and the sum of the angles of a triangle to solve problems involving an unknown angle.

Find the missing angles a , b , c , and d .



2.3 Draw quadrilaterals and triangles from given information about them (e.g., a quadrilateral having equal sides but no right angles, a right isosceles triangle).

Grade Six

Statistics, Data Analysis, and Probability
1.0 Students compute and analyze statistical measurements for data sets:

- 1.1 Compute the range, mean, median, and mode of data sets.
- 1.2 Understand how additional data added to data sets may affect these computations.
- 1.3 Understand how the inclusion or exclusion of outliers affects these computations.
- 1.4 Know why a specific measure of central tendency (mean, median) provides the most useful information in a given context.

2.0 Students use data samples of a population and describe the characteristics and limitations of the samples:

2.1 Compare different samples of a population with the data from the entire population and identify a situation in which it makes sense to use a sample.

2.2 Identify different ways of selecting a sample (e.g., convenience sampling, responses to a survey, random sampling) and which method makes a sample more representative for a population.

- 2.3** Analyze data displays and explain why the way in which the question was asked might have influenced the results obtained and why the way in which the results were displayed might have influenced the conclusions reached.
- 2.4** Identify data that represent sampling errors and explain why the sample (and the display) might be biased.
- 2.5** Identify claims based on statistical data and, in simple cases, evaluate the validity of the claims.

Calvin has been identified as the best runner in your school because he won the 50-yard dash at the all-schools track meet. Use the records of the track team in the table shown below to decide if Calvin is the best runner in the school. Explain your decision, using the data in the table.

Runner	Race 1	Race 2	Race 3	Race 4
Brian	27.3	27.6	30.1	26.2
Maria	26.5	26.3	26.0	27.1
Calvin	30.2	28.1	29.4	25.0
Alice	28.2	29.0	32.0	27.4
Fred	32.1	32.5	29.0	30.0
José	26.2	26.0	25.8	25.5

Soraya has been assigned to do a survey for the student council. However, she forgets to do this task until the morning of the meeting, so she asks three of her best friends what kind of music they would like for a noon-time dance. Their opinions are what Soraya will report to student council.

Do you think Soraya's report is an accurate reflection of the kind of music that students want played for the noon-time dance? Explain your answer.

- 3.0 Students determine theoretical and experimental probabilities and use these to make predictions about events:**
- 3.1** Represent all possible outcomes for compound events in an organized way (e.g., tables, grids, tree diagrams) and express the theoretical probability of each outcome.
- 3.2 Use data to estimate the probability of future events (e.g., batting averages or number of accidents per mile driven).
- 3.3** Represent probabilities as ratios, proportions, decimals between 0 and 1, and percentages between 0 and 100 and verify that the probabilities computed are reasonable; know that if P is the probability of an event, $1-P$ is the probability of an event not occurring.

3.4 Understand that the probability of either of two disjoint events occurring is the sum of the two individual probabilities and that the probability of one event following another, in independent trials, is the product of the two probabilities.

3.5 Understand the difference between independent and dependent events.

Mathematical Reasoning

Grade Six

1.0 Students make decisions about how to approach problems:

- 1.1 Analyze problems by identifying relationships, distinguishing relevant from irrelevant information, identifying missing information, sequencing and prioritizing information, and observing patterns.
- 1.2 Formulate and justify mathematical conjectures based on a general description of the mathematical question or problem posed.
- 1.3 Determine when and how to break a problem into simpler parts.

2.0 Students use strategies, skills, and concepts in finding solutions:

- 2.1 Use estimation to verify the reasonableness of calculated results.
- 2.2 Apply strategies and results from simpler problems to more complex problems.
- 2.3 Estimate unknown quantities graphically and solve for them by using logical reasoning and arithmetic and algebraic techniques.
- 2.4 Use a variety of methods, such as words, numbers, symbols, charts, graphs, tables, diagrams, and models, to explain mathematical reasoning.
- 2.5 Express the solution clearly and logically by using the appropriate mathematical notation and terms and clear language; support solutions with evidence in both verbal and symbolic work.
- 2.6 Indicate the relative advantages of exact and approximate solutions to problems and give answers to a specified degree of accuracy.
- 2.7 Make precise calculations and check the validity of the results from the context of the problem.

3.0 Students move beyond a particular problem by generalizing to other situations:

- 3.1 Evaluate the reasonableness of the solution in the context of the original situation.
- 3.2 Note the method of deriving the solution and demonstrate a conceptual understanding of the derivation by solving similar problems.
- 3.3 Develop generalizations of the results obtained and the strategies used and apply them in new problem situations.

Grade Seven Mathematics Content Standards

By the end of grade seven, students are adept at manipulating numbers and equations and understand the general principles at work. Students understand and use factoring of numerators and denominators and properties of exponents. They know the Pythagorean theorem and solve problems in which they compute the length of an unknown side. Students know how to compute the surface area and volume of basic three-dimensional objects and understand how area and volume change with a change in scale. Students make conversions between different units of measurement. They know and use different representations of fractional numbers (fractions, decimals, and percents) and are proficient at changing from one to another. They increase their facility with ratio and proportion, compute percents of increase and decrease, and compute simple and compound interest. They graph linear functions and understand the idea of slope and its relation to ratio.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified, particularly in the primary grades, before they are used with students.

The symbol ● identifies the key standards for grade seven.

Number Sense

1.0 Students know the properties of, and compute with, rational numbers expressed in a variety of forms:

- 1.1 Read, write, and compare rational numbers in scientific notation (positive and negative powers of 10), compare rational numbers in general.

Put the following numbers on the number line:

$$-3.14 \quad -3.3 \quad -3\frac{1}{3} \quad -3.1 \quad -\frac{27}{8}$$

Arrange the following numbers in increasing order:

$$1.86 \times 10^5 \quad 185,766 \quad 1.004 \times 10^6 \quad 2.1 \times 10^5 \quad 205,666$$

Arrange the following numbers in increasing order:

$$-3.14 \times 10^{-2} \quad 3.14 \times 10^2 \quad -3.14 \times 10^2 \quad 3.14 \times 10^{-2}$$

- 1.2 Add, subtract, multiply, and divide rational numbers (integers, fractions, and terminating decimals) and take positive rational numbers to whole-number powers.

1. $\frac{1}{4} \times 0.33$

2. $\frac{2\frac{1}{7}}{\frac{2}{3}} - \frac{\frac{3}{7}}{\left(\frac{2}{3}\right)^2}$

3. Evaluate:

$$\frac{12}{7} \times \frac{6}{5} \times \frac{7}{8} =$$

$$3\frac{1}{5} \div (-5) =$$

$$(0.2)^5 \times \left(\frac{3}{2}\right)^4 =$$

$$\frac{1}{2} (58.3 - 11.29) =$$

- 1.3 Convert fractions to decimals and percents and use these representations in estimations, computations, and applications.

Change to decimals:

$$\frac{7}{8} \quad \frac{7}{11}$$

- 1.4 Differentiate between rational and irrational numbers.

Which is an irrational number? (CST released test question, 2004)

- (a) $\sqrt{5}$ (b) $\sqrt{9}$ (c) -1 (d) $-\frac{2}{3}$

- 1.5 Know that every rational number is either a terminating or a repeating decimal and be able to convert terminating decimals into reduced fractions.

Change to fractions:

$$0.25 \quad 0.\overline{27}$$

Find the period of the repeating part of $\frac{41}{13}$.

- 1.6 Calculate the percentage of increases and decreases of a quantity.

A sweater originally cost \$37.50. Last week Moesha bought it at 20% off.



How much was deducted from the original price? (CST released test question, 2004)

- (a) \$7.50 (b) \$17.50 (c) \$20.00 (d) \$30.00

- 1.7** Solve problems that involve discounts, markups, commissions, and profit and compute simple and compound interest.

Heather deposits \$800 in an account that earns a flat rate of 10% (simple) interest. Jim deposits \$800 in an account that earns 10% interest compounded yearly. Who will have more money at the end of one year? Two years? Three years? Who will have more money over the long run? Explain why.

Jason bought a jacket on sale for 50% off the original price and another 25% off the discounted price. If the jacket originally cost \$88, what was the final sale price that Jason paid for the jacket? (CST released test question, 2004)

2.0 Students use exponents, powers, and roots and use exponents in working with fractions:

- 2.1 Understand negative whole-number exponents. Multiply and divide expressions involving exponents with a common base.

Simplify:

$$1. \frac{\left(\frac{2}{7}\right)^5 \times \left(\frac{2}{7}\right)^{-11}}{\left(\frac{2}{7}\right)^{-3}}$$

$$2. \left(\frac{2}{3}\right)^{-3} \times \frac{2}{9}$$

- 2.2** Add and subtract fractions by using factoring to find common denominators.

Make use of prime factors to compute:

$$1. \frac{2}{28} + \frac{1}{49}$$

$$2. \frac{-5}{63} + \left(\frac{-7}{99}\right)$$

- 2.3** Multiply, divide, and simplify rational numbers by using exponent rules.

Simplify:

$$1. \frac{\left(-\frac{2}{3}\right)^{-3}}{2\frac{1}{4}} + \left(\frac{3}{-2}\right)^2 \left(4 - 3\frac{1}{3}\right)$$

$$2. \frac{\left(\frac{2}{5} \times 2\frac{1}{3}\right)^4}{\left(\frac{2}{5}\right)\left(-2\frac{1}{3}\right)^3}$$

3. $\frac{3^{-2}}{2^{-3}}$

4. $\frac{2x^3}{2^3x^{-1}}$

5. $\frac{4^2 \cdot 3^5 \cdot 2^4}{4^3 \cdot 3^5 \cdot 2^2}$ (CST released test question, 2004)

- 2.4 Use the inverse relationship between raising to a power and extracting the root of a perfect square integer; for an integer that is not square, determine without a calculator the two integers between which its square root lies and explain why.

Find the length of one side of a square that has an area of 81.

- 2.5** Understand the meaning of the absolute value of a number; interpret the absolute value as the distance of the number from zero on a number line; and determine the absolute value of real numbers.

Is it always true that for any numbers a and b , $a - |b| \leq a + b$?

$|9 - 5| - |6 - 8| = ?$ (CST released test question, 2004)

Grade Seven

Algebra and Functions

1.0 Students express quantitative relationships by using algebraic terminology, expressions, equations, inequalities, and graphs:

- 1.1 Use variables and appropriate operations to write an expression, an equation, an inequality, or a system of equations or inequalities that represents a verbal description (e.g., three less than a number, half as large as area A).

Write the following verbal statements as algebraic expressions:

- The square of a is increased by the sum of twice a and 3.
- The product of $\frac{1}{2}$ of a and 3 is decreased by the quotient of a divided by (-4) .

- 1.2 Use the correct order of operations to evaluate algebraic expressions such as $3(2x + 5)^2$.

Given $x = (-2)$ and $y = 5$ evaluate:

- $x^2 + 2x - 3$
- $\frac{y(xy - 7)}{10}$

- 1.3** Simplify numerical expressions by applying properties of rational numbers (e.g., identity, inverse, distributive, associative, commutative) and justify the process used.

Name the property illustrated by each of the following:

1. $y + -y = 0$

2. $x(y + z) = xy + xz$

3. $x(y + z) = (y + z)x$

4. $x + y = y + x$

5. $y \left(\frac{1}{y} \right) = 1$

- 1.4 Use algebraic terminology (e.g., variable, equation, term, coefficient, inequality, expression, constant) correctly.
- 1.5 Represent quantitative relationships graphically and interpret the meaning of a specific part of a graph in the situation represented by the graph.

2.0 Students interpret and evaluate expressions involving integer powers and simple roots:

- 2.1 Interpret positive whole-number powers as repeated multiplication and negative whole-number powers as repeated division or multiplication by the multiplicative inverse. Simplify and evaluate expressions that include exponents.
- 2.2 Multiply and divide monomials; extend the process of taking powers and extracting roots to monomials when the latter results in a monomial with an integer exponent.

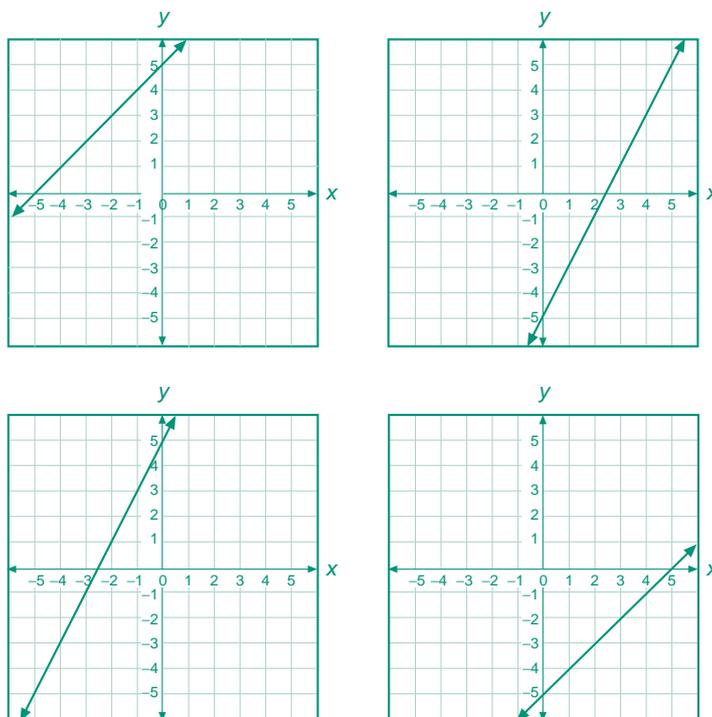
3.0 Students graph and interpret linear and some nonlinear functions:

- 3.1 Graph functions of the form $y = nx^2$ and $y = nx^3$ and use in solving problems.
- 3.2 Plot the values from the volumes of three-dimensional shapes for various values of the edge lengths (e.g., cubes with varying edge lengths or a triangle prism with a fixed height and an equilateral triangle base of varying lengths).

- 3.3** Graph linear functions, noting that the vertical change (change in y -value) per unit of horizontal change (change in x -value) is always the same and know that the ratio (“rise over run”) is called the slope of a graph.

A function of x has value 7 when $x = 1$; it has value 15.5 when $x = 3.5$; and it has value 20 when $x = 5$. Is this a linear function?

Which best represents the graph of $y = 2x - 5$? (CST released test question, 2004)



- 3.4** Plot the values of quantities whose ratios are always the same (e.g., cost to the number of an item, feet to inches, circumference to diameter of a circle). Fit a line to the plot and understand that the slope of the line equals the ratio of the quantities.

4.0 Students solve simple linear equations and inequalities over the rational numbers:

- 4.1** Solve two-step linear equations and inequalities in one variable over the rational numbers, interpret the solution or solutions in the context from which they arose, and verify the reasonableness of the results.

Solve for x if $3x - 12 = 3,821$. If x stands for the number of books in a bookstore, can it satisfy this equation?

What is the solution set to the inequality $6z + 5 > 35$? (CST released test question, 2004)

$$\{z:z < 5\} \quad \{z:z < 24\} \quad \{z:z > 5\} \quad \{z:z > 24\}$$

- 4.2** Solve multistep problems involving rate, average speed, distance, and time or a direct variation.

A train can travel at either of two speeds between two towns that are 72 miles apart. The higher speed is 25% faster than the lower speed and reduces the travel time by 30 minutes. What are the two speeds in miles per hour?

A duck flew at 18 miles per hour for 3 hours, then at 15 miles per hour for 2 hours. How far did the duck fly in all? (CST released test question, 2004)

Juanita earns \$36 for 3 hours of work. At that rate how long would she have to work to earn \$720? (CST released test question, 2004)

Measurement and Geometry

1.0 Students choose appropriate units of measure and use ratios to convert within and between measurement systems to solve problems:

- 1.1 Compare weights, capacities, geometric measures, times, and temperatures within and between measurement systems (e.g., miles per hour and feet per second, cubic inches to cubic centimeters).

Convert the following:

1. 80 miles/hr. = ? ft./sec.
2. 20 oz./min. = ? qts./day

- 1.2 Construct and read drawings and models made to scale.

- 1.3** Use measures expressed as rates (e.g., speed, density) and measures expressed as products (e.g., person-days) to solve problems; check the units of the solutions; and use dimensional analysis to check the reasonableness of the answer.

The chart shown below describes the speed of four printers.

Printer	Description
Roboprint	Prints 2 pages per second
Voltronn	Prints 1 page every 2 seconds
Vantek Plus	Prints 160 pages in 2 minutes
DLS Pro	Prints 100 pages per minute

Which printer is the fastest? (CST released test question, 2004)

2.0 Students compute the perimeter, area, and volume of common geometric objects and use the results to find measures of less common objects. They know how perimeter, area, and volume are affected by changes of scale:

- 2.1 Use formulas routinely for finding the perimeter and area of basic two-dimensional figures and the surface area and volume of basic three-dimensional figures, including rectangles, parallelograms, trapezoids, squares, triangles, circles, prisms, and cylinders.

- 2.2 Estimate and compute the area of more complex or irregular two- and three-dimensional figures by breaking the figures down into more basic geometric objects.
- 2.3 Compute the length of the perimeter, the surface area of the faces, and the volume of a three-dimensional object built from rectangular solids. Understand that when the lengths of all dimensions are multiplied by a scale factor, the surface area is multiplied by the square of the scale factor and the volume is multiplied by the cube of the scale factor.
- 2.4 Relate the changes in measurement with a change of scale to the units used (e.g., square inches, cubic feet) and to conversions between units (1 square foot = 144 square inches or $[1 \text{ ft.}^2] = [144 \text{ in.}^2]$; 1 cubic inch is approximately 16.38 cubic centimeters or $[1 \text{ in.}^3] = [16.38 \text{ cm}^3]$).

Grade Seven

3.0 Students know the Pythagorean theorem and deepen their understanding of plane and solid geometric shapes by constructing figures that meet given conditions and by identifying attributes of figures:

- 3.1 Identify and construct basic elements of geometric figures (e.g., altitudes, midpoints, diagonals, angle bisectors, and perpendicular bisectors; central angles, radii, diameters, and chords of circles) by using a compass and straightedge.
- 3.2 Understand and use coordinate graphs to plot simple figures, determine lengths and areas related to them, and determine their image under translations and reflections.
- 3.3** Know and understand the Pythagorean theorem and its converse and use it to find the length of the missing side of a right triangle and the lengths of other line segments and, in some situations, empirically verify the Pythagorean theorem by direct measurement.

What is the side length of an isosceles right triangle with hypotenuse $\sqrt{72}$?

A right triangle has sides of lengths a , b , and c ; c is the length of the hypotenuse. How would the areas of the three equilateral triangles with sides of lengths a , b , c , respectively, be related to each other?

- 3.4** Demonstrate an understanding of conditions that indicate two geometrical figures are congruent and what congruence means about the relationships between the sides and angles of the two figures.
- 3.5 Construct two-dimensional patterns for three-dimensional models, such as cylinders, prisms, and cones.

- 3.6** Identify elements of three-dimensional geometric objects (e.g., diagonals of rectangular solids) and describe how two or more objects are related in space (e.g., skew lines, the possible ways three planes might intersect).

True or false? If true, give an example. If false, explain why.

Two planes in three-dimensional space can:

1. Intersect in a line.
2. Intersect in a single point.
3. Have no intersection at all.

Statistics, Data Analysis, and Probability

- 1.0** Students collect, organize, and represent data sets that have one or more variables and identify relationships among variables within a data set by hand and through the use of an electronic spreadsheet software program:

- 1.1 Know various forms of display for data sets, including a stem-and-leaf plot or box-and-whisker plot; use the forms to display a single set of data or to compare two sets of data.
- 1.2 Represent two numerical variables on a scatterplot and informally describe how the data points are distributed and any apparent relationship that exists between the two variables (e.g., between time spent on homework and grade level).

- 1.3** Understand the meaning of, and be able to compute, the minimum, the lower quartile, the median, the upper quartile, and the maximum of a data set.

Here is a set of data for an exam in a mathematics class:

Minimum	45
Lower quartile score	51
Median	64
Upper quartile score	72
Maximum	92

1. Suppose there are 15 students in the class. Give a range of scores that would satisfy all the data shown above.
2. Suppose 7 students have scores ranging from 64 to 72. How many students might there be in the class? Explain.

Mathematical Reasoning

1.0 Students make decisions about how to approach problems:

- 1.1 Analyze problems by identifying relationships, distinguishing relevant from irrelevant information, identifying missing information, sequencing and prioritizing information, and observing patterns.
- 1.2 Formulate and justify mathematical conjectures based on a general description of the mathematical question or problem posed.
- 1.3 Determine when and how to break a problem into simpler parts.

Grade Seven

2.0 Students use strategies, skills, and concepts in finding solutions:

- 2.1 Use estimation to verify the reasonableness of calculated results.
- 2.2 Apply strategies and results from simpler problems to more complex problems.
- 2.3 Estimate unknown quantities graphically and solve for them by using logical reasoning and arithmetic and algebraic techniques.
- 2.4 Make and test conjectures by using both inductive and deductive reasoning.
- 2.5 Use a variety of methods, such as words, numbers, symbols, charts, graphs, tables, diagrams, and models, to explain mathematical reasoning.
- 2.6 Express the solution clearly and logically by using the appropriate mathematical notation and terms and clear language; support solutions with evidence in both verbal and symbolic work.
- 2.7 Indicate the relative advantages of exact and approximate solutions to problems and give answers to a specified degree of accuracy.
- 2.8 Make precise calculations and check the validity of the results from the context of the problem.

3.0 Students determine a solution is complete and move beyond a particular problem by generalizing to other situations:

- 3.1 Evaluate the reasonableness of the solution in the context of the original situation.
- 3.2 Note the method of deriving the solution and demonstrate a conceptual understanding of the derivation by solving similar problems.
- 3.3 Develop generalizations of the results obtained and the strategies used and apply them to new problem situations.

Introduction to Grades Eight Through Twelve

The standards for grades eight through twelve are organized differently from those for kindergarten through grade seven. In this section strands are not used for organizational purposes as they are in the elementary grades because the mathematics studied in grades eight through twelve falls naturally under discipline headings: algebra, geometry, and so forth. Many schools teach this material in traditional courses; others teach it in an integrated fashion. To allow local educational agencies and teachers flexibility in teaching the material, the standards for grades eight through twelve do not mandate that a particular discipline be initiated and completed in a single grade. The core content of these subjects must be covered; students are expected to achieve the standards however these subjects are sequenced.

Standards are provided for Algebra I, geometry, Algebra II, trigonometry, mathematical analysis, linear algebra, probability and statistics, advanced placement probability and statistics, and calculus. Many of the more advanced subjects are not taught in every middle school or high school. Moreover, schools and districts have different ways of combining the subject matter in these various disciplines. For example, many schools combine some trigonometry, mathematical analysis, and linear algebra to form a precalculus course. Some districts prefer offering trigonometry content with Algebra II.

Table 1, “Mathematics Disciplines, by Grade Level,” reflects typical grade-level groupings of these disciplines in both integrated and traditional curricula. The lightly shaded region reflects the minimum requirement for mastery by all students. The dark shaded region depicts content that is typically considered elective but that should also be mastered by students who complete the other disciplines in the lower grade levels and continue the study of mathematics.

Many other combinations of these advanced subjects into courses are possible. What is described in this section are standards for the academic content by discipline; this document does not endorse a particular choice of structure for courses or a particular method of teaching the mathematical content.

When students delve deeply into mathematics, they gain not only conceptual understanding of mathematical principles but also knowledge of, and experience with, pure reasoning. One of the most important goals of mathematics is to teach students logical reasoning. The logical reasoning inherent in the study of mathematics allows for applications to a broad range of situations in which answers to practical problems can be found with accuracy.

By grade eight, students’ mathematical sensitivity should be sharpened. Students need to start perceiving logical subtleties and appreciate the need for sound mathematical arguments before making conclusions. Students who are not prepared for Algebra I by grade nine should instead receive specialized instructional materials that focus on the prerequisite standards described in Appendix E. An algebra readiness course will prepare students for success in algebra and subsequent advanced courses. As students progress in the study of mathematics, they learn to distinguish between inductive and deductive reasoning; understand

Table I. Mathematics Disciplines, by Grade Level

Disciplines	Grades				
	Eight	Nine	Ten	Eleven	Twelve
Algebra I					
Geometry					
Algebra II					
Probability and Statistics					
Trigonometry					
Linear Algebra					
Mathematical Analysis					
Advanced Placement Probability and Statistics					
Calculus					

the meaning of logical implication; test general assertions; realize that one counterexample is enough to show that a general assertion is false; understand conceptually that although a general assertion is true in a few cases, it may not be true in all cases; distinguish between something being proven and a mere plausibility argument; and identify logical errors in chains of reasoning.

Mathematical reasoning and conceptual understanding are not separate from content; they are intrinsic to the mathematical discipline students master at more advanced levels.

Algebra I Mathematics Content Standards

Symbolic reasoning and calculations with symbols are central in algebra. Through the study of algebra, a student develops an understanding of the symbolic language of mathematics and the sciences. In addition, algebraic skills and concepts are developed and used in a wide variety of problem-solving situations.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students identify and use the arithmetic properties of subsets of integers and rational, irrational, and real numbers, including closure properties for the four basic arithmetic operations where applicable:

1.1 Students use properties of numbers to demonstrate whether assertions are true or false.

2.0 Students understand and use such operations as taking the opposite, finding the reciprocal, taking a root, and raising to a fractional power. They understand and use the rules of exponents.

Simplify $(x^3 y^{1/2})^6 \sqrt{xy}$.

3.0 Students solve equations and inequalities involving absolute values.

Solve for x : $3|x| + 5 > 7$.

For which values of x is $|x + 4| = |x| + 4$?

4.0 Students simplify expressions before solving linear equations and inequalities in one variable, such as $3(2x - 5) + 4(x - 2) = 12$.

For what values of x is the following inequality valid?

$5(x - 1) > 3x + 2$.

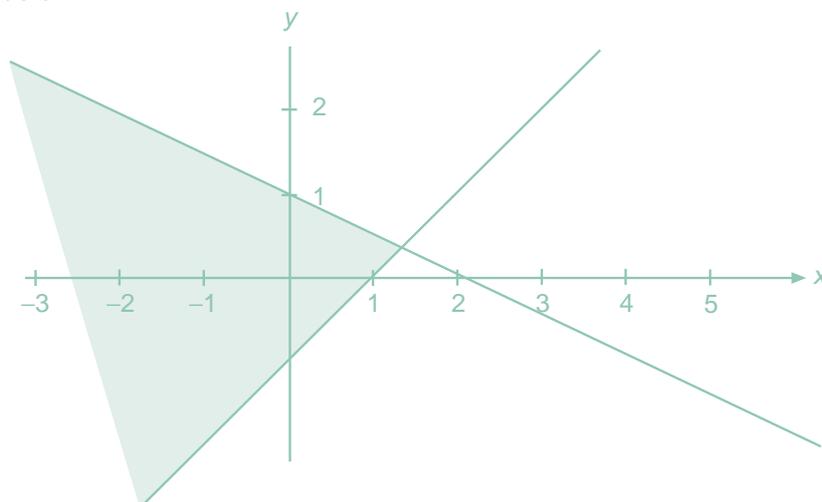
Expand and simplify $2(3x + 1) - 8x$.

5.0 Students solve multistep problems, including word problems, involving linear equations and linear inequalities in one variable and provide justification for each step.

A-1 Pager Company charges a \$25 set-up fee plus a \$6.50 monthly charge. Cheaper Beeper charges \$8 per month with no set-up fee. Set up an inequality to determine how long one would need to have the pager until the A-1 Pager plan would be the less expensive one.

- 6.0** Students graph a linear equation and compute the x - and y -intercepts (e.g., graph $2x + 6y = 4$). They are also able to sketch the region defined by linear inequalities (e.g., they sketch the region defined by $2x + 6y < 4$).

Find inequalities whose simultaneous solution defines the region shown below:



Algebra I

- 7.0** Students verify that a point lies on a line, given an equation of the line. Students are able to derive linear equations by using the point-slope formula.

Does the point $(1, 2)$ lie on, above, or below the graph of the line $3x - 5y + 8 = 0$? Explain how you can be sure of your answer.

Write the equation of the line having x -intercept $-2\frac{1}{3}$ and y -intercept 5.

- 8.0** Students understand the concepts of parallel lines and perpendicular lines and how their slopes are related. Students are able to find the equation of a line perpendicular to a given line that passes through a given point.

Find the equation of the line passing through $(-1, \frac{1}{3})$ and parallel to the line defined by $5x + 2y = 17$. Also find the equation of the line passing through the same point but perpendicular to the line $5x + 2y = 17$.

- 9.0** Students solve a system of two linear equations in two variables algebraically and are able to interpret the answer graphically. Students are able to solve a system of two linear inequalities in two variables and to sketch the solution sets.

Solve and sketch the lines and the solution set:

$$3x + y = -1$$

$$x - \frac{1}{2}y = \frac{4}{3}$$

-
- 10.0** Students add, subtract, multiply, and divide monomials and polynomials. Students solve multistep problems, including word problems, by using these techniques.
-
- 11.0** Students apply basic factoring techniques to second- and simple third-degree polynomials. These techniques include finding a common factor for all terms in a polynomial, recognizing the difference of two squares, and recognizing perfect squares of binomials.
- Factor $9x^3 + 6x^2 + x$.
-
- 12.0** Students simplify fractions with polynomials in the numerator and denominator by factoring both and reducing them to the lowest terms.
- Simplify $\frac{x^2 + 2x + 1}{x^2 - 1}$.
-
- 13.0** Students add, subtract, multiply, and divide rational expressions and functions. Students solve both computationally and conceptually challenging problems by using these techniques.
- Solve for x and give a reason for each step: $\frac{2}{3x+1} + 2 = \frac{2}{3}$. (ICAS 1997, 6)
-
- 14.0** Students solve a quadratic equation by factoring or completing the square.
-
- 15.0** Students apply algebraic techniques to solve rate problems, work problems, and percent mixture problems.
- The sum of the two digits of a number is 10. If 36 is added to it, the digits will be reversed. Find the number.
- Two cars A and B move at constant velocity. Car A starts from P to Q , 150 miles apart, at the same time that car B starts from Q to P . They meet at the end of $1\frac{1}{2}$ hours. If car A moves 10 miles per hour faster than car B, what are their velocities?
-
- 16.0** Students understand the concepts of a relation and a function, determine whether a given relation defines a function, and give pertinent information about given relations and functions.
-
- 17.0** Students determine the domain of independent variables and the range of dependent variables defined by a graph, a set of ordered pairs, or a symbolic expression.
-
- 18.0** Students determine whether a relation defined by a graph, a set of ordered pairs, or a symbolic expression is a function and justify the conclusion.

- 19.0** Students know the quadratic formula and are familiar with its proof by completing the square.

Toni is solving this equation by completing the square.

$$ax^2 + bx + c = 0 \text{ (where } a \geq 0\text{)}$$

Step 1. $ax^2 + bx = -c$

Step 2. $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Step 3. ?

Which response shown below should be step 3 in the solution?

1. $x^2 = -\frac{c}{b} - \frac{b}{a}x$.

2. $x + \frac{b}{a} = -\frac{c}{ax}$.

3. $x^2 + \frac{b}{a}x + \frac{b}{2a} = -\frac{c}{a} + \frac{b}{2a}$.

4. $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$.

(CST released test question, 2004)

- 20.0** Students use the quadratic formula to find the roots of a second-degree polynomial and to solve quadratic equations.

Suppose the graph of $y = px^2 + 5x + 2$ intersects the x -axis at two distinct points, where p is a constant. What are the possible values of p ?

- 21.0** Students graph quadratic functions and know that their roots are the x -intercepts.

The graph of $y = x^2 + bx - 1$ passes through $(-\frac{1}{3}, 0)$

What is b ?

- 22.0** Students use the quadratic formula or factoring techniques or both to determine whether the graph of a quadratic function will intersect the x -axis in zero, one, or two points.

- 23.0** Students apply quadratic equations to physical problems, such as the motion of an object under the force of gravity.

-
- 24.0** Students use and know simple aspects of a logical argument:
- 24.1 Students explain the difference between inductive and deductive reasoning and identify and provide examples of each.
 - 24.2 Students identify the hypothesis and conclusion in logical deduction.
 - 24.3 Students use counterexamples to show that an assertion is false and recognize that a single counterexample is sufficient to refute an assertion.
-
- 25.0** Students use properties of the number system to judge the validity of results, to justify each step of a procedure, and to prove or disprove statements:
- 25.1 Students use properties of numbers to construct simple, valid arguments (direct and indirect) for, or formulate counterexamples to, claimed assertions.
 - 25.2 Students judge the validity of an argument according to whether the properties of the real number system and the order of operations have been applied correctly at each step.
 - 25.3 Given a specific algebraic statement involving linear, quadratic, or absolute value expressions or equations or inequalities, students determine whether the statement is true sometimes, always, or never.

Geometry Mathematics Content Standards

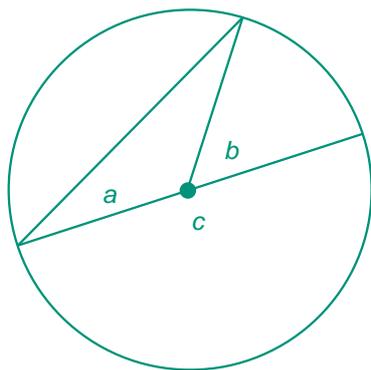
Chapter 2
Mathematics
Content
Standards

The geometry skills and concepts developed in this discipline are useful to all students. Aside from learning these skills and concepts, students will develop their ability to construct formal, logical arguments and proofs in geometric settings and problems.

1.0 Students demonstrate understanding by identifying and giving examples of undefined terms, axioms, theorems, and inductive and deductive reasoning.

2.0 Students write geometric proofs, including proofs by contradiction. If a line L is tangent to a circle at a point P , prove that the radius passing through P is perpendicular to L .

If C is the center of the circle in the figure shown below, prove that angle b has twice the measure of angle a .



Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

3.0 Students construct and judge the validity of a logical argument and give counterexamples to disprove a statement.

Prove or disprove: If two triangles have two pairs of congruent sides, the triangles must be congruent.

4.0 Students prove basic theorems involving congruence and similarity.

Prove that in a triangle, the larger angle faces the longer side.

If L_1 , L_2 , and L_3 are three parallel lines such that the distance from L_1 to L_2 is equal to the distance from L_2 to L_3 , and if l is any transversal that intersects L_1 , L_2 , and L_3 at A_1 , A_2 , and A_3 , respectively, prove that the segments A_1A_2 and A_2A_3 are congruent.

- 5.0** Students prove that triangles are congruent or similar, and they are able to use the concept of corresponding parts of congruent triangles.

Prove that a quadrilateral that has two pairs of congruent opposite angles is a parallelogram.

Prove that in $\triangle ABC$, if D is the midpoint of side AB and a line passing through D and parallel to BC intersects side AC at E , then E is the midpoint of side AC .

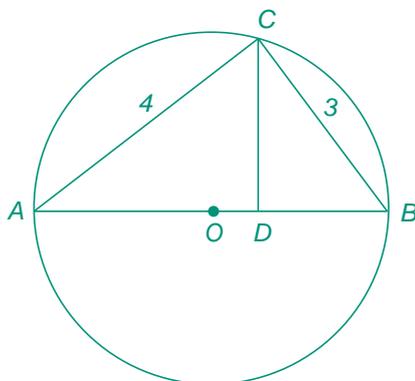
- 6.0** Students know and are able to use the triangle inequality theorem.

- 7.0** Students prove and use theorems involving the properties of parallel lines cut by a transversal, the properties of quadrilaterals, and the properties of circles.

Prove that the figure formed by joining, in order, the midpoints of the sides of a quadrilateral is a parallelogram.

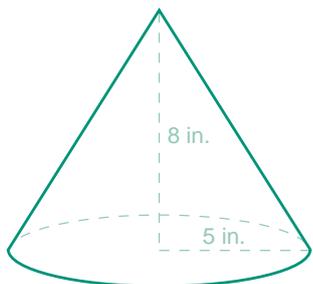
Using what you know about parallel lines cut by a transversal, show that the sum of the angles in a triangle is the same as the angle in a straight line, 180 degrees.

AB is a diameter of a circle centered at O . $CD \perp AB$. If the length of AB is 5, find the length of side CD .



- 8.0** Students know, derive, and solve problems involving the perimeter, circumference, area, volume, lateral area, and surface area of common geometric figures.

A right circular cone has radius 5 inches and height 8 inches.

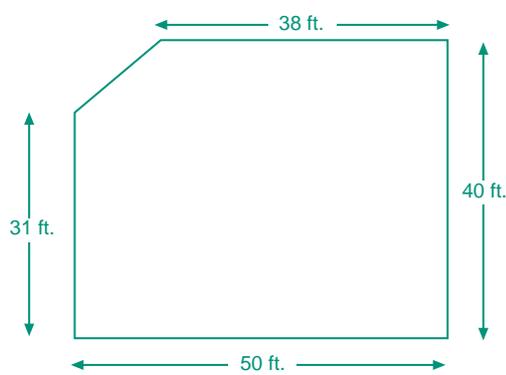


What is the lateral area of the cone? (Lateral area of cone = πrl , where l = slant height.) (CST released test question, 2004)

- 9.0** Students compute the volumes and surface areas of prisms, pyramids, cylinders, cones, and spheres; and students commit to memory the formulas for prisms, pyramids, and cylinders.

- 10.0** Students compute areas of polygons, including rectangles, scalene triangles, equilateral triangles, rhombi, parallelograms, and trapezoids.

The diagram below shows the overall floor plan for a house. It has right angles at three corners. What is the area of the house? What is the perimeter of the house? (CERT 1997, 26)



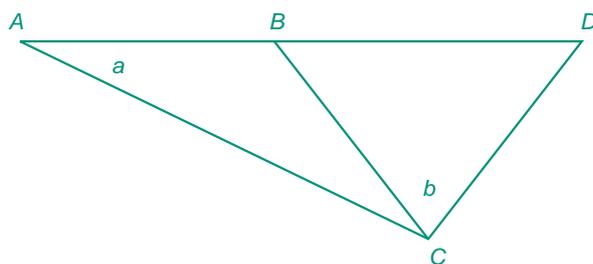
- 11.0** Students determine how changes in dimensions affect the perimeter, area, and volume of common geometric figures and solids.

A triangle has sides of lengths a , b , and c and an area A . What is the area of a triangle with sides of lengths $3a$, $3b$, and $3c$, respectively? Prove that your answer is correct.

- 12.0** Students find and use measures of sides and of interior and exterior angles of triangles and polygons to classify figures and solve problems.

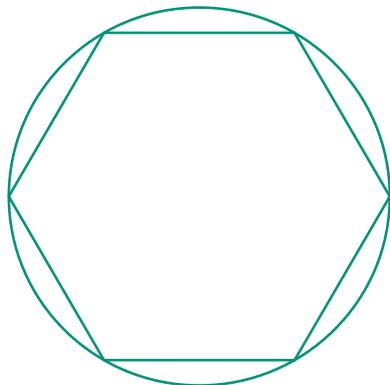
- 13.0** Students prove relationships between angles in polygons by using properties of complementary, supplementary, vertical, and exterior angles.

In the figure below, $\overline{AB} = \overline{BC} = \overline{CD}$. Find an expression for the measure of angle b in terms of the measure of angle a and prove that your expression is correct.



-
- 14.0** Students prove the Pythagorean theorem.
-
- 15.0** Students use the Pythagorean theorem to determine distance and find missing lengths of sides of right triangles.
-
- 16.0** Students perform basic constructions with a straightedge and compass, such as angle bisectors, perpendicular bisectors, and the line parallel to a given line through a point off the line.
- Prove that the standard construction of the perpendicular from a point to a line not containing the point is correct.
-
- 17.0** Students prove theorems by using coordinate geometry, including the midpoint of a line segment, the distance formula, and various forms of equations of lines and circles.
- Use coordinates to prove that if ABC is a triangle and D, E are points on sides AB and AC , respectively, so that
- $$\frac{|AD|}{|AB|} = \frac{|AE|}{|AC|},$$
- then line DE is parallel to BC .
-
- 18.0** Students know the definitions of the basic trigonometric functions defined by the angles of a right triangle. They also know and are able to use elementary relationships between them. For example, $\tan(x) = \sin(x)/\cos(x)$, $(\sin(x))^2 + (\cos(x))^2 = 1$.
- Without using a calculator, determine which is larger, $\tan(60^\circ)$ or $\tan(70^\circ)$ and explain why.
-
- 19.0** Students use trigonometric functions to solve for an unknown length of a side of a right triangle, given an angle and a length of a side.
-
- 20.0** Students know and are able to use angle and side relationships in problems with special right triangles, such as 30° , 60° , and 90° triangles and 45° , 45° , and 90° triangles.
-
- 21.0** Students prove and solve problems regarding relationships among chords, secants, tangents, inscribed angles, and inscribed and circumscribed polygons of circles.

Use the perimeter of a regular hexagon inscribed in a circle to explain why $\pi > 3$. (ICAS 1997,11)⁴



Chapter 2
Mathematics
Content
Standards

Geometry

22.0 Students know the effect of rigid motions on figures in the coordinate plane and space, including rotations, translations, and reflections.

Use rigid motions to prove the side-angle-side criterion of triangle congruence.

⁴ The Web site showing the source for the problems from the Intersegmental Committee of the Academic Senates (ICAS) is in the "Web Resources" section in "Works Cited."

Algebra II Mathematics Content Standards

This discipline complements and expands the mathematical content and concepts of Algebra I and geometry. Students who master Algebra II will gain experience with algebraic solutions of problems in various content areas, including the solution of systems of quadratic equations, logarithmic and exponential functions, the binomial theorem, and the complex number system.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

1.0 Students solve equations and inequalities involving absolute value.

Sketch the graph of each function.

$$y = \left| \frac{1}{x} \right|.$$

$$y = -\frac{2}{3} |x - 2| - 5.$$

2.0 Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or with matrices.

Draw the region in the plane that is the solution set for the inequality $(x - 1)(x + 2y) > 0$.

3.0 Students are adept at operations on polynomials, including long division.

Divide $x^4 - 3x^2 + 3x$ by $x^2 + 2$, and write the answer in the form:

$$\text{polynomial} + \frac{\text{linear polynomial}}{x^2 + 2}.$$

4.0 Students factor polynomials representing the difference of squares, perfect square trinomials, and the sum and difference of two cubes.

Factor $x^3 + 8$.

5.0 Students demonstrate knowledge of how real and complex numbers are related both arithmetically and graphically. In particular, they can plot complex numbers as points in the plane.

6.0 Students add, subtract, multiply, and divide complex numbers.

Write $\frac{1+i}{1-2i}$ in the form of $a + bi$, where a and b are real numbers.

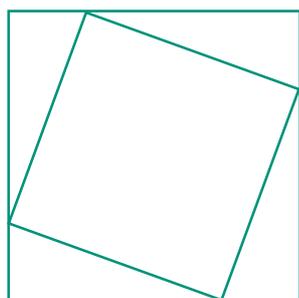
- 7.0** Students add, subtract, multiply, divide, reduce, and evaluate rational expressions with monomial and polynomial denominators and simplify complicated rational expressions, including those with negative exponents in the denominator.

Simplify $\frac{(x^2 - x)^2}{x(x-1)^{-2}(x^2 + 3x - 4)}$.

Algebra II

- 8.0** Students solve and graph quadratic equations by factoring, completing the square, or using the quadratic formula. Students apply these techniques in solving word problems. They also solve quadratic equations in the complex number system.

In the figure shown below, the area between the two squares is 11 square inches. The sum of the perimeters of the two squares is 44 inches. Find the length of a side of the larger square. (ICAS 1997, 12)



- 9.0** Students demonstrate and explain the effect that changing a coefficient has on the graph of quadratic functions; that is, students can determine how the graph of a parabola changes as a , b , and c vary in the equation $y = a(x - b)^2 + c$.

- 10.0** Students graph quadratic functions and determine the maxima, minima, and zeros of the function.

Find a quadratic function of x that has zeros at $x = -1$ and $x = 2$. Find a cubic equation of x that has zeros at $x = -1$ and $x = 2$ and nowhere else. (ICAS 1997, 7)

- 11.0** Students prove simple laws of logarithms.

- 11.1** Students understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Solve: $2^x = 5(13^{2x-5})$.

11.2 Students judge the validity of an argument according to whether the properties of real numbers, exponents, and logarithms have been applied correctly at each step.

Algebra II

12.0 Students know the laws of fractional exponents, understand exponential functions, and use these functions in problems involving exponential growth and decay.

The number of bacteria in a colony was growing exponentially. At 1 p.m. yesterday the number of bacteria was 100, and at 3 p.m. yesterday it was 4,000. How many bacteria were there in the colony at 6 p.m. yesterday? (TIMSS gr.12, K-13)

13.0 Students use the definition of logarithms to translate between logarithms in any base.

14.0 Students understand and use the properties of logarithms to simplify logarithmic numeric expressions and to identify their approximate values.

1. Find the largest integer that is less than:

$$\log_{10} (1,256)$$

$$\log_{10} (.029)$$

2. $\frac{1}{2} \log_2 64 = ?$

15.0 Students determine whether a specific algebraic statement involving rational expressions, radical expressions, or logarithmic or exponential functions is sometimes true, always true, or never true.

For positive numbers x and y , is the equation $\log_2 xy = \log_2 x \cdot \log_2 y$ always true, sometimes true, or never true?

If c is a real number, for what values of c is it true that $\frac{\sqrt{(c^2 - 1)^4}}{c + 1} = c - 1$?

16.0 Students demonstrate and explain how the geometry of the graph of a conic section (e.g., asymptotes, foci, eccentricity) depends on the coefficients of the quadratic equation representing it.

What is the graph of $x^2 + py^2 - 4x + 10y - 26 = 0$ when $p = 1$?

When $p = 4$? When $p = -4$?

17.0 Given a quadratic equation of the form $ax^2 + by^2 + cx + dy + e = 0$, students can use the method for completing the square to put the equation into standard form and can recognize whether the graph of the equation is a circle, ellipse, parabola, or hyperbola. Students can then graph the equation.

Does the origin lie inside, outside, or on the geometric figure whose equation is $x^2 + y^2 - 10x + 10y - 1 = 0$? Explain your reasoning. (ICAS 1997, 11)

18.0 Students use fundamental counting principles to compute combinations and permutations.

19.0 Students use combinations and permutations to compute probabilities.

20.0 Students know the binomial theorem and use it to expand binomial expressions that are raised to positive integer powers.

What is the third term of $(2x - 1)^6$? What is the general term?
What is a simplified expression for the sum?

21.0 Students apply the method of mathematical induction to prove general statements about the positive integers.

Use mathematical induction to prove that for any integer $n \geq 1$, $1 + 3 + 5 + \dots + (2n - 1) = n^2$.

22.0 Students find the general term and the sums of arithmetic series and of both finite and infinite geometric series.

Find the sum of the arithmetic series: $13 + 16 + 19 + \dots + 94$.

Find the sum of the geometric series:

$$\frac{3^5}{5^2} + \frac{3^6}{5^3} + \frac{3^7}{5^4} + \dots + \frac{3^{32}}{5^{29}}$$

23.0 Students derive the summation formulas for arithmetic series and for both finite and infinite geometric series.

24.0 Students solve problems involving functional concepts, such as composition, defining the inverse function and performing arithmetic operations on functions.

Which of the following functions are their own inverse functions? Use at least two different methods to answer this question and explain your methods:

$$f(x) = \frac{2}{x} \quad g(x) = x^3 + 4 \quad h(x) = \frac{2 + \ln(x)}{2 - \ln(x)} \quad j(x) = \sqrt[3]{\frac{x^3 + 1}{x^3 - 1}}$$

(ICAS 1997, 13)

25.0 Students use properties from number systems to justify steps in combining and simplifying functions.

Trigonometry Mathematics Content Standards

Trigonometry uses the techniques that students have previously learned from the study of algebra and geometry. The trigonometric functions studied are defined geometrically rather than in terms of algebraic equations. Facility with these functions as well as the ability to prove basic identities regarding them is especially important for students intending to study calculus, more advanced mathematics, physics and other sciences, and engineering in college.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

-
- 1.0** Students understand the notion of angle and how to measure it, in both degrees and radians. They can convert between degrees and radians.
-
- 2.0** Students know the definition of sine and cosine as y - and x -coordinates of points on the unit circle and are familiar with the graphs of the sine and cosine functions.
- Find an angle β between 0 and 2π such that $\cos(\beta) = \cos(6\pi/7)$ and $\sin(\beta) = -\sin(6\pi/7)$. Find an angle θ between 0 and 2π such that $\sin(\theta) = \cos(6\pi/7)$ and $\cos(\theta) = \sin(6\pi/7)$.
-
- 3.0** Students know the identity $\cos^2(x) + \sin^2(x) = 1$:
- 3.1 Students prove that this identity is equivalent to the Pythagorean theorem (i.e., students can prove this identity by using the Pythagorean theorem and, conversely, they can prove the Pythagorean theorem as a consequence of this identity).
- 3.2 Students prove other trigonometric identities and simplify others by using the identity $\cos^2(x) + \sin^2(x) = 1$. For example, students use this identity to prove that $\sec^2(x) = \tan^2(x) + 1$.
- Prove $\csc^2 x = 1 + \cot^2 x$.
-
- 4.0** Students graph functions of the form $f(t) = A \sin(Bt + C)$ or $f(t) = A \cos(Bt + C)$ and interpret A , B , and C in terms of amplitude, frequency, period, and phase shift.
- On a graphing calculator, graph the function $f(x) = \sin(x) \cos(x)$. Select a window so that you can carefully examine the graph.
1. What is the apparent period of this function?
 2. What is the apparent amplitude of this function?
 3. Use this information to express f as a simpler trigonometric function.
-
- 5.0** Students know the definitions of the tangent and cotangent functions and can graph them.

6.0 Students know the definitions of the secant and cosecant functions and can graph them.

7.0 Students know that the tangent of the angle that a line makes with the x -axis is equal to the slope of the line.

8.0 Students know the definitions of the inverse trigonometric functions and can graph the functions.

Trigonometry

9.0 Students compute, by hand, the values of the trigonometric functions and the inverse trigonometric functions at various standard points.

10.0 Students demonstrate an understanding of the addition formulas for sines and cosines and their proofs and can use those formulas to prove and/or simplify other trigonometric identities.

Use the addition formula for sine to find a numerical value of $\sin(75^\circ)$.

Use the addition formula to find the numerical value of $\sin(15^\circ)$.

Is $g(x) = 5 \sin 3x + 2 \cos x$ a periodic function? If so, what is its period? What is its amplitude?

11.0 Students demonstrate an understanding of half-angle and double-angle formulas for sines and cosines and can use those formulas to prove and/or simplify other trigonometric identities.

Express $\sin 3x$ in terms of $\sin x$ and $\cos x$.

12.0 Students use trigonometry to determine unknown sides or angles in right triangles.

13.0 Students know the law of sines and the law of cosines and apply those laws to solve problems.

A vertical pole sits between two points that are 60 feet apart. Guy wires to the top of that pole are staked at the two points. The guy wires are 40 feet and 35 feet long. How tall is the pole?

14.0 Students determine the area of a triangle, given one angle and the two adjacent sides.

Suppose in $\triangle ABC$ and $\triangle A'B'C'$, the sides of AB and $A'B'$ are congruent, as are AC and $A'C'$, but $\angle A$ is bigger than $\angle A'$. Which of $\triangle ABC$ and $\triangle A'B'C'$ has a bigger area? Prove that your answer is correct.

15.0 Students are familiar with polar coordinates. In particular, they can determine polar coordinates of a point given in rectangular coordinates and vice versa.

-
- 16.0** Students represent equations given in rectangular coordinates in terms of polar coordinates.

Express the circle of radius 2 centered at (2, 0) in polar coordinates.

- 17.0** Students are familiar with complex numbers. They can represent a complex number in polar form and know how to multiply complex numbers in their polar form.

What is the angle that the ray from the origin to $3 + \sqrt{3}i$ makes with the positive x -axis?

- 18.0** Students know DeMoivre's theorem and can give n th roots of a complex number given in polar form.
-

- 19.0** Students are adept at using trigonometry in a variety of applications and word problems.

A lighthouse stands 100 feet above the surface of the ocean. From what distance can it be seen? (Assume that the radius of the earth is 3,960 miles.)

Mathematical Analysis Mathematics Content Standards

This discipline combines many of the trigonometric, geometric, and algebraic techniques needed to prepare students for the study of calculus and strengthens their conceptual understanding of problems and mathematical reasoning in solving problems. These standards take a functional point of view toward those topics. The most significant new concept is that of limits. Mathematical analysis is often combined with a course in trigonometry or perhaps with one in linear algebra to make a yearlong precalculus course.

-
- 1.0** Students are familiar with, and can apply, polar coordinates and vectors in the plane. In particular, they can translate between polar and rectangular coordinates and can interpret polar coordinates and vectors graphically.
-
- 2.0** Students are adept at the arithmetic of complex numbers. They can use the trigonometric form of complex numbers and understand that a function of a complex variable can be viewed as a function of two real variables. They know the proof of DeMoivre's theorem.
-
- 3.0** Students can give proofs of various formulas by using the technique of mathematical induction.
- Use mathematical induction to show that the sum of the interior angles in a convex polygon with n sides is $(n - 2) \cdot 180^\circ$.
-
- 4.0** Students know the statement of, and can apply, the fundamental theorem of algebra.
- Find all cubic polynomials of x that have zeros at $x = -1$ and $x = 2$ and nowhere else. (ICAS 1997, 13)
-
- 5.0** Students are familiar with conic sections, both analytically and geometrically:
- 5.1 Students can take a quadratic equation in two variables; put it in standard form by completing the square and using rotations and translations, if necessary; determine what type of conic section the equation represents; and determine its geometric components (foci, asymptotes, and so forth).
- 5.2 Students can take a geometric description of a conic section—for example, the locus of points whose sum of its distances from $(1, 0)$ and $(-1, 0)$ is 6—and derive a quadratic equation representing it.
-
- 6.0** Students find the roots and poles of a rational function and can graph the function and locate its asymptotes.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

-
- 7.0** Students demonstrate an understanding of functions and equations defined parametrically and can graph them.
-
- 8.0** Students are familiar with the notion of the limit of a sequence and the limit of a function as the independent variable approaches a number or infinity. They determine whether certain sequences converge or diverge.

Linear Algebra Mathematics Content Standards

The general goal in this discipline is for students to learn the techniques of matrix manipulation so that they can solve systems of linear equations in any number of variables. Linear algebra is most often combined with another subject, such as trigonometry, mathematical analysis, or precalculus.

- | | |
|------|---|
| 1.0 | Students solve linear equations in any number of variables by using Gauss-Jordan elimination. |
| 2.0 | Students interpret linear systems as coefficient matrices and the Gauss-Jordan method as row operations on the coefficient matrix. |
| 3.0 | Students reduce rectangular matrices to row echelon form. |
| 4.0 | Students perform addition on matrices and vectors. |
| 5.0 | Students perform matrix multiplication and multiply vectors by matrices and by scalars. |
| 6.0 | Students demonstrate an understanding that linear systems are inconsistent (have no solutions), have exactly one solution, or have infinitely many solutions. |
| 7.0 | Students demonstrate an understanding of the geometric interpretation of vectors and vector addition (by means of parallelograms) in the plane and in three-dimensional space. |
| 8.0 | Students interpret geometrically the solution sets of systems of equations. For example, the solution set of a single linear equation in two variables is interpreted as a line in the plane, and the solution set of a two-by-two system is interpreted as the intersection of a pair of lines in the plane. |
| 9.0 | Students demonstrate an understanding of the notion of the inverse to a square matrix and apply that concept to solve systems of linear equations. |
| 10.0 | Students compute the determinants of 2×2 and 3×3 matrices and are familiar with their geometric interpretations as the area and volume of the parallelepipeds spanned by the images under the matrices of the standard basis vectors in two-dimensional and three-dimensional spaces. |
| 11.0 | Students know that a square matrix is invertible if, and only if, its determinant is nonzero. They can compute the inverse to 2×2 and 3×3 matrices using row reduction methods or Cramer's rule. |
| 12.0 | Students compute the scalar (dot) product of two vectors in n -dimensional space and know that perpendicular vectors have zero dot product. |

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

Probability and Statistics Mathematics Content Standards

This discipline is an introduction to the study of probability, interpretation of data, and fundamental statistical problem solving. Mastery of this academic content will provide students with a solid foundation in probability and facility in processing statistical information.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

-
- 1.0** Students know the definition of the notion of *independent events* and can use the rules for addition, multiplication, and complementation to solve for probabilities of particular events in finite sample spaces.
-
- 2.0** Students know the definition of *conditional probability* and use it to solve for probabilities in finite sample spaces.
- A whole number between 1 and 30 is chosen at random. If the digits of the number that is chosen add up to 8, what is the probability that the number is greater than 12?
-
- 3.0** Students demonstrate an understanding of the notion of *discrete random variables* by using them to solve for the probabilities of outcomes, such as the probability of the occurrence of five heads in 14 coin tosses.
-
- 4.0** Students are familiar with the standard distributions (normal, binomial, and exponential) and can use them to solve for events in problems in which the distribution belongs to those families.
-
- 5.0** Students determine the mean and the standard deviation of a normally distributed random variable.
-
- 6.0** Students know the definitions of the *mean*, *median*, and *mode* of a distribution of data and can compute each in particular situations.
-
- 7.0** Students compute the variance and the standard deviation of a distribution of data.
- Find the mean and standard deviation of the following seven numbers:
4 12 5 6 8 5 9
- Make up another list of seven numbers with the same mean and a smaller standard deviation. Make up another list of seven numbers with the same mean and a larger standard deviation. (ICAS 1997, 11)
-
- 8.0** Students organize and describe distributions of data by using a number of different methods, including frequency tables, histograms, standard line and bar graphs, stem-and-leaf displays, scatterplots, and box-and-whisker plots.

Advanced Placement Probability and Statistics

Mathematics Content Standards

Chapter 2 Mathematics Content Standards

This discipline is a technical and in-depth extension of probability and statistics. In particular, mastery of academic content for advanced placement gives students the background to succeed in the *Advanced Placement* examination in the subject.

1.0 Students solve probability problems with finite sample spaces by using the rules for addition, multiplication, and complementation for probability distributions and understand the simplifications that arise with independent events.

2.0 Students know the definition of *conditional probability* and use it to solve for probabilities in finite sample spaces.

You have 5 coins in your pocket: 1 penny, 2 nickels, 1 dime, and 1 quarter. If you pull out 2 coins at random and they are collectively worth more than 10 cents, what is the probability that you pulled out a quarter?

3.0 Students demonstrate an understanding of the notion of *discrete random variables* by using this concept to solve for the probabilities of outcomes, such as the probability of the occurrence of five or fewer heads in 14 coin tosses.

4.0 Students understand the notion of a *continuous random variable* and can interpret the probability of an outcome as the area of a region under the graph of the probability density function associated with the random variable.

Consider a continuous random variable X whose possible values are numbers between 0 and 2 and whose probability density function is given by $f(x) = 1 - \frac{1}{2}x$ for $0 \leq x \leq 2$. What is the probability that $X > 1$?

5.0 Students know the definition of the *mean of a discrete random variable* and can determine the mean for a particular discrete random variable.

6.0 Students know the definition of the *variance of a discrete random variable* and can determine the variance for a particular discrete random variable.

7.0 Students demonstrate an understanding of the standard distributions (normal, binomial, and exponential) and can use the distributions to solve for events in problems in which the distribution belongs to those families.

Suppose that X is a normally distributed random variable with mean $m = 0$. If $P(X < c) = \frac{2}{3}$, find $P(-c < X < c)$.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

-
- 8.0** Students determine the mean and the standard deviation of a normally distributed random variable.
-
- 9.0** Students know the central limit theorem and can use it to obtain approximations for probabilities in problems of finite sample spaces in which the probabilities are distributed binomially.
-
- 10.0** Students know the definitions of the *mean*, *median*, and *mode of distribution* of data and can compute each of them in particular situations.
-
- 11.0** Students compute the variance and the standard deviation of a distribution of data.
-
- 12.0** Students find the line of best fit to a given distribution of data by using least squares regression.
-
- 13.0** Students know what the *correlation coefficient of two variables* means and are familiar with the coefficient's properties.
-
- 14.0** Students organize and describe distributions of data by using a number of different methods, including frequency tables, histograms, standard line graphs and bar graphs, stem-and-leaf displays, scatterplots, and box-and-whisker plots.
-
- 15.0** Students are familiar with the notions of a statistic of a distribution of values, of the sampling distribution of a statistic, and of the variability of a statistic.
-
- 16.0** Students know basic facts concerning the relation between the mean and the standard deviation of a sampling distribution and the mean and the standard deviation of the population distribution.
-
- 17.0** Students determine confidence intervals for a simple random sample from a normal distribution of data and determine the sample size required for a desired margin of error.
-
- 18.0** Students determine the *P*-value for a statistic for a simple random sample from a normal distribution.
-
- 19.0** Students are familiar with the *chi-square* distribution and *chi-square* test and understand their uses.
-

Calculus Mathematics Content Standards

When taught in high school, calculus should be presented with the same level of depth and rigor as are entry-level college and university calculus courses. These standards outline a complete college curriculum in one-variable calculus. Many high school programs may have insufficient time to cover all of the following content in a typical academic year. For example, some districts may treat differential equations lightly and spend substantial time on infinite sequences and series. Others may do the opposite. Consideration of the College Board syllabi for the Calculus AB and Calculus BC sections of the *Advanced Placement Examinations in Mathematics* may be helpful in making curricular decisions. Calculus is a widely applied area of mathematics and involves a beautiful intrinsic theory. Students mastering this content will be exposed to both aspects of the subject.

- 1.0** Students demonstrate knowledge of both the formal definition and the graphical interpretation of limit of values of functions. This knowledge includes one-sided limits, infinite limits, and limits at infinity. Students know the definition of convergence and divergence of a function as the domain variable approaches either a number or infinity:
- 1.1 Students prove and use theorems evaluating the limits of sums, products, quotients, and composition of functions.
 - 1.2 Students use graphical calculators to verify and estimate limits.
 - 1.3 Students prove and use special limits, such as the limits of $(\sin(x))/x$ and $(1 - \cos(x))/x$ as x tends to 0.

Evaluate the following limits, justifying each step:

$$\lim_{x \rightarrow 4} \frac{x - 4}{\sqrt{x} - 2}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{\sin(3x)}$$

$$\lim_{x \rightarrow \infty} \left(x - \sqrt{x^2 - x} \right)$$

- 2.0** Students demonstrate knowledge of both the formal definition and the graphical interpretation of continuity of a function.

For what values of x is the function $f(x) = \frac{x^2 - 1}{x^2 - 4x + 3}$ continuous? Explain.

Note: The sample problems illustrate the standards and are written to help clarify them. Some problems are written in a form that can be used directly with students; others will need to be modified before they are used with students.

-
- 3.0** Students demonstrate an understanding and the application of the intermediate value theorem and the extreme value theorem.
-
- 4.0** Students demonstrate an understanding of the formal definition of the derivative of a function at a point and the notion of differentiability:
- 4.1 Students demonstrate an understanding of the derivative of a function as the slope of the tangent line to the graph of the function.
- 4.2 Students demonstrate an understanding of the interpretation of the derivative as an instantaneous rate of change. Students can use derivatives to solve a variety of problems from physics, chemistry, economics, and so forth that involve the rate of change of a function.
- 4.3 Students understand the relation between differentiability and continuity.
- 4.4 Students derive derivative formulas and use them to find the derivatives of algebraic, trigonometric, inverse trigonometric, exponential, and logarithmic functions.
- Find all points on the graph of $f(x) = \frac{x^2 - 2}{x + 1}$ where the tangent line is parallel to the tangent line at $x = 1$.
-
- 5.0** Students know the chain rule and its proof and applications to the calculation of the derivative of a variety of composite functions.
-
- 6.0** Students find the derivatives of parametrically defined functions and use implicit differentiation in a wide variety of problems in physics, chemistry, economics, and so forth.
- For the curve given by the equation $\sqrt{x} + \sqrt{y} = 4$, use implicit differentiation to find $\frac{d^2y}{dx^2}$.
-
- 7.0** Students compute derivatives of higher orders.
-
- 8.0** Students know and can apply Rolle's theorem, the mean value theorem, and L'Hôpital's rule.
-
- 9.0** Students use differentiation to sketch, by hand, graphs of functions. They can identify maxima, minima, inflection points, and intervals in which the function is increasing and decreasing.
-
- 10.0** Students know Newton's method for approximating the zeros of a function.

- 11.0** Students use differentiation to solve optimization (maximum-minimum problems) in a variety of pure and applied contexts.

A man in a boat is 24 miles from a straight shore and wishes to reach a point 20 miles down shore. He can travel 5 miles per hour in the boat and 13 miles per hour on land. Find the minimal time for him to reach his destination and where along the shore he should land the boat to arrive as fast as possible.

Calculus

- 12.0** Students use differentiation to solve related rate problems in a variety of pure and applied contexts.

- 13.0** Students know the definition of the definite integral by using Riemann sums. They use this definition to approximate integrals.

The following is a Riemann sum that approximates the area under the graph of a function $f(x)$, between $x = a$ and $x = b$. Determine a possible formula for the function $f(x)$ and for the values of a and b : $\sum_{i=1}^n \frac{2}{n} e^{1+\frac{2i}{n}}$.

- 14.0** Students apply the definition of the integral to model problems in physics, economics, and so forth, obtaining results in terms of integrals.

- 15.0** Students demonstrate knowledge and proof of the fundamental theorem of calculus and use it to interpret integrals as antiderivatives.

If $f(x) = \int_1^x \sqrt{1+t^3} dt$, find $f'(2)$.

- 16.0** Students use definite integrals in problems involving area, velocity, acceleration, volume of a solid, area of a surface of revolution, length of a curve, and work.

- 17.0** Students compute, by hand, the integrals of a wide variety of functions by using techniques of integration, such as substitution, integration by parts, and trigonometric substitution. They can also combine these techniques when appropriate.

Evaluate the following:

$$\int \frac{\sin(1-\sqrt{x})}{\sqrt{x}} dx$$

$$\int_1^e \frac{\ln x}{\sqrt{x}} dx$$

$$\int_0^1 \sqrt{1+\sqrt{x}} dx$$

$$\int \arctan x dx$$

$$\int \frac{\sqrt{x^2-1}}{x^3} dx$$

$$\int \frac{dx}{e^x \sqrt{1-e^{2x}}}$$

18.0 Students know the definitions and properties of inverse trigonometric functions and the expression of these functions as indefinite integrals.

19.0 Students compute, by hand, the integrals of rational functions by combining the techniques in standard 17.0 with the algebraic techniques of partial fractions and completing the square.

20.0 Students compute the integrals of trigonometric functions by using the techniques noted above.

21.0 Students understand the algorithms involved in Simpson's rule and Newton's method. They use calculators or computers or both to approximate integrals numerically.

22.0 Students understand improper integrals as limits of definite integrals.

23.0 Students demonstrate an understanding of the definitions of convergence and divergence of sequences and series of real numbers. By using such tests as the comparison test, ratio test, and alternate series test, they can determine whether a series converges.

Determine whether the following alternating series converge absolutely, converge conditionally, or diverge:

$$\sum_{n=3}^{\infty} (-1)^n \left(\frac{2^n}{n!} \right) \quad \sum_{n=3}^{\infty} \frac{(-1)^n}{n \ln n} \quad \sum_{n=3}^{\infty} (-1)^n \left(\frac{1+n}{n + \ln n} \right)$$

24.0 Students understand and can compute the radius (interval) of the convergence of power series.

25.0 Students differentiate and integrate the terms of a power series in order to form new series from known ones.

26.0 Students calculate Taylor polynomials and Taylor series of basic functions, including the remainder term.

27.0 Students know the techniques of solution of selected elementary differential equations and their applications to a wide variety of situations, including growth-and-decay problems.